Self-simulability of right-angled Artin groups

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May 20, 2026

Abstract

A group is self-simulable if all its computable actions admit SFT covers, which means roughly that they can be implemented with finitely many tiling constraints. We prove that a RAAG is self-simulable if and only if its defining graph has no disconnecting clique, and also prove partial results on self-simulability of general graph products.

1 Introduction

Let Γ be a group, and A a finite set, called the *alphabet*. The main object of study in the field of symbolic dynamics is the *subshift*, namely a topologically closed and Γ-invariant subsystem of the translation action $\Gamma \curvearrowright A^\Gamma$. Our convention for the translation action is that a group Γ acts on the left on a subshift $\mathcal{S} \subset A^\Gamma$ by $\Gamma \times \mathcal{S} \ni (g, c) \mapsto (h \mapsto c(g^{-1}h))$.

An interesting class of subshifts are *subshifts of finite type* or *SFTs*, obtained by removing from A^Γ those points whose Γ-orbit intersects a clopen set (this can be seen as "forbidding finitely many patterns").

A factor map (surjective shift-equivariant continuous map) from an SFT to a Γ-system is known as an *SFT cover*, and we call systems having an SFT cover *SFT covered*. Finding SFT covers is a standard method of studying hyperbolic toral automorphisms [9, 17] as well as the boundaries of hyperbolic groups [12]. SFT covered subshifts are called *sofic shifts* [22], and they are in themselves an interesting class of dynamical systems.

An SFT cover can be seen as a form of finite presentation of the system. Namely, an SFT is fully described by a clopen set which can be described by a finite amount of data if Γ is finitely-generated. In many cases also the factor map admits a finite description. In particular, this happens in the case of sofic shifts, where the Curtis-Hedlund-Lyndon theorem [11] provides a combinatorial characterization of factor maps. More generally finite descriptions of the factor map exist for all expansive factors [13], and for many non-expansive ones [16].

SFT covered systems are typically a large class of dynamical systems. Hochman showed in [15] a strong theorem, which we state in simplified form:

Theorem (Hochman). *Let $\mathbb{Z}^d \curvearrowright X$ be any effective action, where X is an effectively closed subset of Cantor space. Then the $\mathbb{Z}^{d+2}$-system on X where the two new generators act trivially, is SFT covered.*

Here an *effectively closed* subset of Cantor space $\{0,1\}^\omega$ refers to a subset obtained by removing a countable set of cylinders enumerated by a Turing machine (that is, a Π_1^0 set), and an *effective action* is one where each group element acts by a computable homeomorphism.

By an example of Jeandel (see [5]), the statement of Hochman's theorem fails to hold if $\mathbb{Z}^{d+2}$ is replaced by $\mathbb{Z}^{d+1}$.

Hochman's theorem suggests the following definition:

Definition 1. *Suppose $\phi : G \to H$ is a surjective group homomorphism. To every H-system, we may associate its* pullback (along ϕ)*, namely the G-system by defining $g \cdot x = \phi(g) \cdot x$. We say that G* simulates H (through ϕ) *if every pullback of an effective H-system along ϕ is SFT covered.*

Plenty of examples of this phenomenon are known, for G, G_1, G_2, G_3, H infinite finitely-generated groups with decidable word problem, H non-amenable, and the maps ϕ the obvious ones:

- $\mathbb{Z}^{d+2}$ simulates $\mathbb{Z}^d$ [15] (the theorem above),
- $\mathbb{Z}^d \rtimes G$ for $d \geq 2$ simulates G for any semidirect product [3],
- $G_1 \times G_2 \times G_3$ simulates each of the G_i [2],
- $G_1 \times H$ simulates H [4],
- $\mathbb{Z}_2 \wr \mathbb{Z}$ simulates $\mathbb{Z}$ [7].

Simulation theorems have many consequences. For example, they typically allow the construction of strongly aperiodic subshifts of finite type (i.e. SFTs where every orbit is free). The construction of such SFTs is a common theme in the field, starting with the classical construction of Berger on $\mathbb{Z}^2$ [8]. For several groups (including Thompson's V or the Grigorchuk group), the only known constructions of strongly aperiodic SFTs come from simulation theorems. On amenable groups, simulation theorems can also be used to obtain SFTs with arbitrary Π_1^0 entropies (this requires some control on the fibers in the simulation, but it does happens in the simulations above).

An interesting case is when a group simulates itself (through the identity map), called *self-simulability*. It was shown in [1] that this property implies nonamenability and one-endedness. Further, a geometric obstruction to self-simulation named *UFOs* is presented in [6]. This obstruction amounts to a thin subset (typically, an amenable subgroup) separating the group. We make extensive use of these results, and in particular of the following lemma.

Lemma 1.1. *Let $\Gamma_1, \Gamma_2, \Delta$ be finitely generated groups with Δ amenable and $\iota_1 : \Delta \to \Gamma_1, \iota_2 : \Delta \to \Gamma_2$ proper embeddings. Then the amalgamated free product $\Gamma_1 *_{\iota_1,\iota_2} \Gamma_2$ is not self-simulable.*

In [5], it was shown that for groups with decidable word problem, self-simulability is equivalent to all effective subshifts on the group being sofic. The first examples were also given of self-simulable groups in [5]. For example, the following groups are self-simulable

- $G \times H$ where both G, H are f.g. non-amenable groups,
- Thompson's V,
- $\mathrm{GL}(n, \mathbb{Z})$ for $n \geq 5$,

- braid groups with at least 7 braids,
- all non-amenable branch groups,
- certain right-angled Artin groups.

The case $G = H = F_2$ of the first item is the prototypical example of a self-simulable group, and the other items are deduced from this. The results of this paper generalize the first item by providing an almost complete characterization of the graph products of infinite groups that are self-simulable. In particular, we characterize self-simulable right-angled Artin groups. The results of this paper, however, do not follow from a direct application of the results in [5] and require a new construction.

1.1 Graph Products

Recall that if (V, E) is a finite simplicial graph where to each $u \in V$ we have associated a group G_u. We assume that the G_u are disjoint, apart from sharing the identity element. Then the corresponding *graph product* is

$$\Gamma(G) = \langle \bigcup_{u \in V} G_u \mid [G_u, G_v] \text{ when } (u, v) \in E \rangle,$$

i.e. this is can be thought of as the freest group where each G_u $(u \in V)$ embeds, and for each edge $(u, v) \in E$, the corresponding groups G_u, G_v commute.

We note that the definition above makes sense even if the graph (V, E) is infinite. However, we recall that the graph product is a finitely-generated group if and only if the graph (V, E) is finite, and all the groups G_u are finitely-generated. As explained in [5, Section 3.2], an infinitely-generated group is never self-simulable. Thus, we restrict our attention to finite graphs.

We will sometimes confuse the vertices V with the corresponding groups, and more generally sets of vertices with the subgroups generated by the corresponding vertex groups. For example, an *amenable clique* is a clique in (V, E) to every vertex of which is associated an amenable group. Note that a clique corresponds to a direct product, and a direct product of amenable groups is amenable, so indeed the subgroup generated by amenable vertex groups in a clique is itself amenable.

The *right-angled Artin groups* or *RAAGs* are the graph products where all the G_u are infinite cyclic, that is $G_u \cong \mathbb{Z}$. The *right-angled Coxeter groups* or *RACGs* are the graph products where all the G_u are cyclic of order 2, that is $G_u \cong \mathbb{Z}_2$.

If the word problem is decidable in each G_u, then it is also decidable in the graph product. In particular, the word problem is decidable on RAAGs and RACGs.

We obtain the following criterion for self-simulability of graph products.

Let us say a set of nodes $C \subset V$ in a graph (V, E) is *disconnecting* if the number of connected components in $(V \setminus C, E \cap (V \setminus C)^2)$ is not equal to 1. Thus, C is disconnecting if either $C = V$, or there exist $a, b \notin C$ such that there is no path from a to b that does not enter C. (It may not seem natural to include the case $C = V$, but this is the more convenient choice for us.)

Theorem 1. *Let $G = (V, E, (G_u)_{u \in V})$ be a finite graph every node of which is a finitely-generated and infinite group with decidable word problem. Assume G is not a clique with exactly one non-amenable node. Then the following conditions are equivalent:*

1. *G has no disconnecting amenable clique.*
2. *The graph product $\Gamma(G)$ is self-simulable.*

Note that the previously known examples of self-simulable groups stem from the simple case where G is a clique on two non-amenable nodes.

The proof of self-simulability of the groups generalizes the proof of self-simulability of $F_2 \times F_2$ by using the normal form for graph products to find suitable "directions" where information should be stored.

The case of a clique with a single non-amenable node of course is not characterized by the same condition. For instance, when G has just one vertex u, whose vertex group G_u is non-amenable, we have $\Gamma(G) \cong G_u$, and of course the self-simulability is not dictated by the graph (but instead by whether G_u is self-simulable). Even if we know the self-simulation status of the vertex groups G_u, giving a full characterization of self-simulability of graph products would require solving the following question from [5]:

Question 1. *Let Γ, Δ be finitely-generated groups. If $\Gamma \times \Delta$ is self-simulable and Δ is amenable, is Γ necessarily self-simulable?*

Indeed, a clique with a single non-amenable vertex has as a graph product $\Gamma \times \Delta$ where Γ is the non-amenable vertex and Δ is amenable. Hence, a positive answer to Question 1 would ensure that the self-simulability of the graph product is entirely determined by the self-simulability of Γ.

The assumption that the node groups are infinite is also essential for the method, and the general case stays wide open.

As a corollary, we obtain a complete characterization of self-simulable RAAGs.

The result in [5] about RAAGs is the following:

Theorem. *Suppose $G = (V, E)$ is a finite connected graph which has two edges with the property that no $v \in V$ is adjacent to both of them. Then the RAAG corresponding to the complement graph of (V, E) is self-simulable.*

Question 9.8 of the same paper leaves open the full characterization of such RAAGs. In the case of RAAGs, we get a complete characterization as a consequence of Theorem 1.

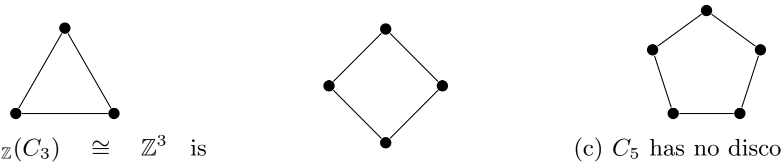
Theorem 2. *Let $G = (V, E)$ a finite simplicial graph. The following conditions are equivalent.*

1. *G has no disconnecting clique.*
2. *The right-angled Artin group $\Gamma_{\mathbb{Z}}(G)$ is self-simulable.*

Again, we recall that we consider a clique graph to be a disconnecting clique in itself.

It is shown in [23, 19] that the existence of a disconnecting clique can be determined in polynomial time. Thus, our theorem reduces the question of self-simulability to an easily checkable purely graph-theoretic condition.

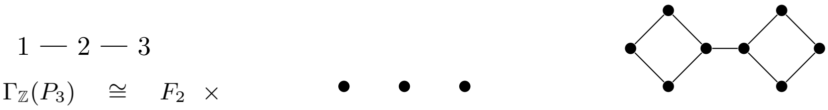
Example 1: RAAGs of trees are never self-simulable as they are either cliques (with at most two nodes), or they are disconnected by any one of their inner vertices. RAAGs of cycles of length at least 4 are always self-simulable as any disconnecting vertex set must contain two non-adjacent vertices. Some more examples of applying this theorem can be found in Figure 1. ○



(a) $\Gamma_{\mathbb{Z}}(C_3) \cong \mathbb{Z}^3$ is amenable, so not self-simulable by [5]. $C = C_3$ a disconnecting clique.

(b) $\Gamma_{\mathbb{Z}}(C_4) \cong F_2 \times F_2$ is self-simulable [5]. There is no disconnecting clique.

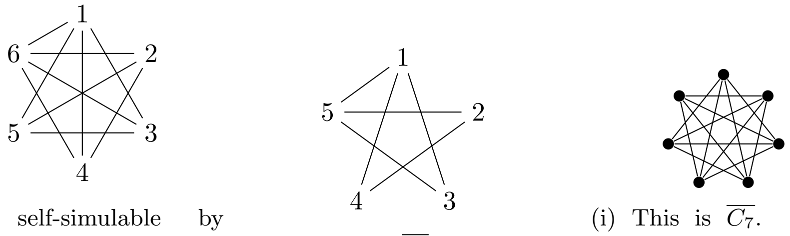
(c) C_5 has no disconnecting clique so the RAAG is self-simulable. (This case was open.)



(d) $\Gamma_{\mathbb{Z}}(P_3) \cong F_2 \times \mathbb{Z}$ is a product of a multi-ended group and an amenable group, so not self-simulable [5]; $\{2\}$ is a disconnecting clique.

(e) $\Gamma_{\mathbb{Z}}(3K_1) \cong F_3$ is multi-ended, so not self-simulable [5], $\emptyset$ (or any vertex) is a disconnecting clique.

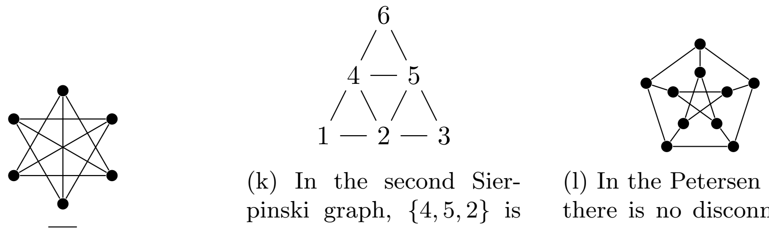
(f) Disconnecting edge, so the RAAG $F_2^2 *_{\mathbb{Z}^2} F_2^2$ is not self-simulable. Follows also from Lemma 1.1 above (from [6]).



(g) self-simulable by Corollary 8.13 in [5] (in the complement P_6, $d(\{1,2\},\{5,6\}) > 2$). No disconnecting clique.

(h) This is $\overline{P_5}$. $\{1,5\}$ is a disconnecting clique, so the RAAG $F_2^2 *_{\mathbb{Z}^2} \mathbb{Z}^3$ is not self-simulable.

(i) This is $\overline{C_7}$. By an ad hoc proof [5, Example 8.15], the RAAG is self-simulable. No disconnecting clique.



(j) This is $\overline{C_6}$. Results of [5] do not apply. No disconnecting clique.

(k) In the second Sierpinski graph, $\{4,5,2\}$ is a disconnecting clique, so the RAAG is not self-simulable.

(l) In the Petersen graph, there is no disconnecting clique so the RAAG is self-simulable. Results of [5] do not apply.

Figure 1: Examples of applying Theorem 2 to various RAAGs.

One direction of Theorem 1 is almost clear from Lemma 1.1. Indeed, we have

Corollary 1.1.1. *Let $G = (V, E)$ be a finite simplicial graph, $C \subset G$ a disconnecting clique of G, and $(\Gamma_v)_{v \in G}$ groups. If for all $v \in V(C), \Gamma_v$ is amenable,* *then the graph product* $\Gamma(G)$ *is not self-simulable.*

Proof. Let G_1, G_2 be two subgraphs of G verifying that $G_1, G_2 \neq C, G_1 \cup G_2 = G$ and $G_1 \cap G_2 = C$. Then, we may write $\Gamma(G) = \Gamma(G_1) *_{\Gamma(C)} \Gamma(G_2)$. But since C is a clique with amenable vertices, $\Gamma(C)$ is a direct product of amenable groups, and so it is amenable. We conclude with Lemma 1.1. ☐

2 Convention

$$0 \in \mathbb{N}$$

3 Self-simulable graph products of infinite groups

The goal of this section will be to prove Theorems 1 and 2. For the sake of brevity, we introduce the following definition.

Definition 2 (Atomic Graph)**.** *Let* $(A_i)_{i\in I}, (B_j)_{j\in J}$ *two families of f.g. groups such that the* A_i *are infinite and amenable and the* B_j *are non-amenable. Let* $G = (V, E)$ *a graph with* $V = \{A_i \mid i \in I\} \cup \{B_j \mid j \in J\}$. G *is atomic if:*

- *It admits no disconnecting clique the vertices of which are amenable.*
- $|J| \geq 2$ *or* G *does not form a clique.*

One implication of Theorem 1 then reduces to showing that graph products of atomic graphs are self-simulable.

As a warmup, we begin with an outline of the argument in the case of RAAGs. Hence, let G be a simplicial graph that has no disconnecting clique, and let $\Gamma(G) \curvearrowright X$ be an effective action.

1. We construct a subshift of finite type that does the following. First, at every point g of the group, we pick a set of directions $\mathbf{B}(g)$, that is a set of vertices of G. The coset $g\langle\mathbf{B}(g)\rangle$ is called the bush at g. We ask that the bush contains a coset that is isomorphic to $\mathbb{Z}^2$. We further ask that the bushes at g and at gs have a direction in common that commutes with s. Finally, we ask that the set of vertices corresponding to each bush forms a connected subgraph of G. In the case of RAAGs, simply choosing for $\mathbf{B}(g)$ the complement of the possible last letters of a reduced writing of g satisfies the conditions.

2. We then choose a direction in each bush on which to write an element of $\{0, 1\}^{\mathbb{N}}$. The point now is to make sure that this element is in X, and that for every generator s, the element that is written on the bush at gs is obtained by applying s^{-1} to the element that is written on the bush at g. Indeed, if this is the case, the map that sends a configuration of the subshift to the element that is written on the bush at $1_{\Gamma_{\mathbb{Z}}(G)}$ will be equivariant.

3. Since the set of directions chosen at g forms a connected graph, we can make sure that the word written on every direction beginning at g is the same. This is done by synchronizing the word along the diagonals of each embedded $\mathbb{Z}^2$ in the bush. This is shown in Figure 2.

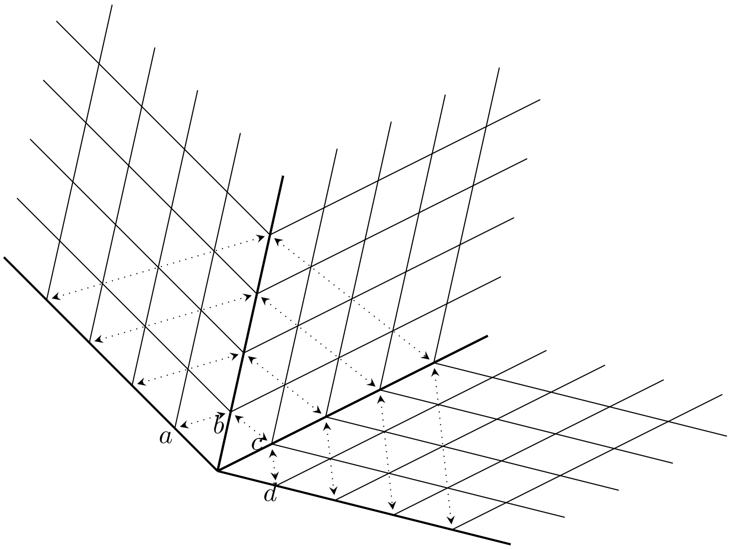


Figure 2: Suppose the bush at node g is $\{a, b, c, d\}$, and $\{a, b\}, \{b, c\}, \{c, d\}$ commute pairwise. Each commuting pair spans a grid, and we use the dotted diagonal lines on the grids to synchronize Y-configurations stored on the rays $g\{s^n \mid n \in \mathbb{N}\}$ for different values of $s \in \{a, b, c, d\}$.

4. Now, we use each bush $g\langle \mathbf{B}(g) \rangle$ to simulate a Turing machine, by inscribing an instance of the tiling problem on a copy of $\mathbb{Z}^2$ that is contained in the bush. This Turing machine checks that the element of $\{0, 1\}^{\mathbb{N}}$ that is written on $g\langle \mathbf{B}(g) \rangle$ is in X, and that the element that is written on $gs\langle \mathbf{B}(gs) \rangle$ is obtained by applying s. This can be done, because the bushes $g\langle \mathbf{B}(g) \rangle$ and $gs\langle \mathbf{B}(gs) \rangle$ share a direction, say a, that commutes with s. Hence, the cosets $g\langle a \rangle$ and $gs\langle a \rangle$ stay at bounded distance from one another.[1] This is illustrated in Figure 3.

5. Finally, we check that the map that sends a configuration to the element of $\{0, 1\}^{\mathbb{N}}$ that is written at $1_{\Gamma_{\mathbb{Z}}(G)}$ is surjective onto X, and so it is a factor map. This is done by checking that the bushes can be chosen so that if $g\langle \mathbf{B}(g) \rangle = h\langle \mathbf{B}(h) \rangle$ and $\mathbf{B}(g) = \mathbf{B}(h)$, then $g = h$. Indeed, if this is the case then the bushes of different elements g using the same set of directions are pairwise disjoint, so for each g, the bush at g may be used entirely for the computation of g.

We are now ready for the proof in the general case of a graph product. The main difference between a RAAG and a general amenable group is that (because the group might not contain any elements of infinite order) we need to replace the straight lines $\langle a \rangle$ by paths that are chosen by an SFT (this is indeed possible, using the fact that every infinite group admits a translation-like action of $\mathbb{Z}$ [18]).

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[1] A technical detail is that to be able to perform the comparison easily, we need to compute not only a single $x \in X$, but also its neighbors, see Definition 3.

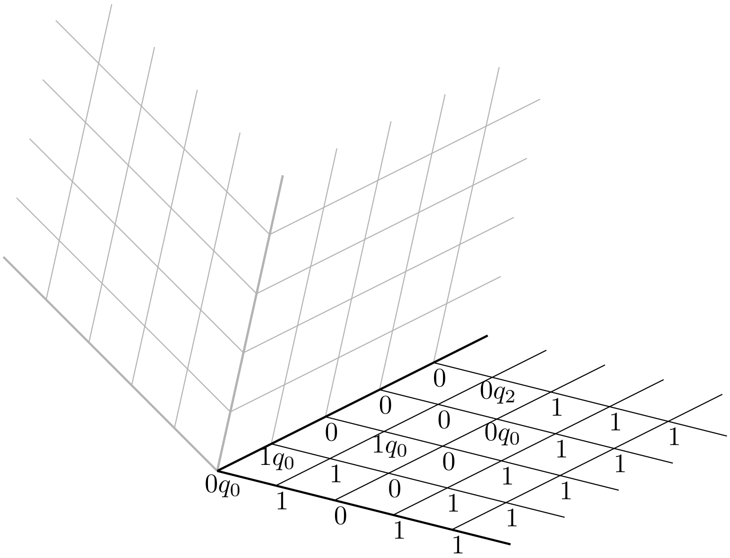

Figure 3: One of the grids is used for computation, here we use $\{gc^md^n \mid m,n \in \mathbb{N}\}$ with d as the "space direction" and c as the "time direction". The example Turing machine shown is just an adding machine over a binary alphabet, in the construction we replace this by the machine that never halts if and only if the configuration is in Y.

In the amenable case, when we use a direction at g, we will consider the entire bush (the coset $g\mathbf{B}(g)$) "used", and ensure that other elements g have essentially disjoint bushes (more precisely, we will allow a bounded number of reuses). In the case of a nonamenable group, a paradoxical decomposition of a group allows every node to have a separate path. We can use an SFT to mark a paradoxical decomposition as in [5].

We begin with some elementary graph theoretic facts.

Lemma 3.1. *If G is an atomic graph and $C \subset G$ is an amenable clique which does not contain all elements of G, then $G \setminus C$ is connected, every point of C is connected to a point of $G \setminus C$ and $G \setminus C$ contains at least two points.*

Proof. The connectedness of $G \setminus C$ is a direct consequence of the definition. If there exists a point $p \in C$ that is connected to no point of $G \setminus C$, then $C \setminus \{p\}$ is a disconnecting clique. Finally, if $|J| \geq 2$, it is clear that $G \setminus C \supset \{B_j \mid j \in J\}$ contains at least two points, and if the $\{A_i \mid i \in I\}$ do not form a clique, then there must exist $A_{i_0} \notin C$. But every point of C is connected to a point of $G \setminus C$, so that A_{i_0} cannot be alone in $G \setminus C$ without $\{A_i \mid i \in I\}$ being a clique. $\square$

Let us assume $G = (V = I \cup J, E)$, with G_i amenable for every $i \in I$, and G_j non-amenable for every $j \in J$. For $v \in V$, denote $\text{link}(v)$ its link, i.e. $\text{link}(v) = \{u \in V \mid \{u,v\} \in E\}$. Note that $v \notin \text{link}(v)$.

Let $K_v \Subset G_v$. Define for all $i \in I, \Sigma_i = K_i^2 \times \{\mathbf{b}\}$, for all $j \in J, \Sigma_j = K_j^3 \times \{\mathbf{r}, \mathbf{c}\}$ and $\Sigma = (\prod_{i \in I} \Sigma_i) \times (\prod_{j \in J} \Sigma_j)$. For each $u \in V$, write $\text{col} : \Sigma_u \to \{\mathbf{b}, \mathbf{r}, \mathbf{c}\}$ the projection on the last coordinate, and $\overline{\mathtt{t}}$ the opposite color of color $\mathtt{t}$ (that is $\overline{\mathbf{r}} = \mathbf{c}, \overline{\mathbf{c}} = \mathbf{r}$ and $\overline{\mathbf{b}} = \mathbf{b}$). Let $S_v \Subset G_v$ finite generating sets of the G_v. Let $S = \bigcup_{v \in v} S_v$.

Define the path subshift $\mathcal{P}$ on alphabet Σ by demanding the following conditions of every $((\rho_i = ((\ell_i, r_i), \mathbf{b})_{i \in I}, (\rho_j = ((\ell_j{}^{\mathbf{r}}, \ell_j{}^{\mathbf{c}}, r_j), c_j))_{j \in J}) \in \mathcal{P}$ at every $g \in \Gamma$.

1. For every $i \in I, r_i(g\ell_i(g)) = \ell_i(g)^{-1}$.
2. For every $i \in I, \ell_i(gr_i(g)) = r_i(g)^{-1}$.
3. For every $j \in J, c_j(g\ell_j{}^{\mathbf{r}}(g))) = \mathbf{r}$ and $r_j(g\ell_j{}^{\mathbf{r}}(g)) = \ell_j{}^{\mathbf{r}}(g)^{-1}$.
4. For every $j \in J, c_j(g\ell_j{}^{\mathbf{c}}(g))) = \mathbf{c}$ and $r_j(g\ell_j{}^{\mathbf{c}}(g)) = \ell_j{}^{\mathbf{c}}(g)^{-1}$.
5. For every $j \in J, \ell_j^{c_j(g)}(gr_j(g)) = r_j(g)^{-1}$.
6. For every $\{u, v\} \in E$, for every $a \in S_v, \rho_u(g) = \rho_u(ga)$.

It is clear that $\mathcal{P}$ is of finite type.

For $\rho \in \mathcal{P}$ and $v \in V$, we define a path by following the left edges of the opposite color of a vertex:

$$\gamma_g^v(n, \rho) = \begin{cases} 1_\Gamma & \text{if } n = 0 \\ \gamma_g^v(n-1, \rho)\ell_v^{\overline{c_v(g)}}(g\gamma_g^v(n-1, \rho)) & \text{otherwise.} \end{cases}$$

Note that for every $h \in G_v$,

$$\gamma_{h^{-1}}^v(n, \rho) = \gamma_{1_\Gamma}^v(n, h\rho).$$

This is a simple calculation, but one may also avoid the calculation with the correct mental picture: If we visualize the configurations on the right Cayley graph, the shift by h moves the vertex 1_Γ to h, and carries the configuration by the unique graph automorphism. The equality comes from that fact that the definition of $\gamma_g^v(n, \rho)$ can be seen in terms of local movement of a "reading head" on the configuration from initial position g. The configuration (n, ρ) relative to position h^{-1} and the configuration $(n, h\rho)$ relative to position 1_Γ are the same, by the definition of the shift map.

Note also that for every $u \in \text{link}(v), h \in G_u$,

$$\gamma_{1_\Gamma}^v(n, \rho) = \gamma_{1_\Gamma}^v(n, h\rho).$$

This in turn follows by an easy induction from the last item of the definition of the subshift.

Claim 3.1. *For a suitable choice of the $K_v \Subset G_v$, the path subshift contains a configuration ρ that is such that for every $v \in V$, and for every $g \in \Gamma$, the path $(n \in \mathbb{N} \mapsto \gamma_g^v(n, \rho))$ is injective.*

Proof. For every $v \in V$, denote π_v the canonical projection $\Gamma(G) \to G_v$ defined by mapping $\pi_v(g) = g$ for $g \in G_v$, $\pi_v(g) = 1_{\Gamma(G)}$ for $g \in G_u$ when $u \neq v$, and extending uniquely. This gives a well-defined homomorphism, and of course the restriction $\pi_v|G_v : G_v \to G_v$ is the identity map.

Seward showed in [18, Theorem 4.1] that every finitely generated infinite group G has a translation like action of $\mathbb{Z}$, i.e. an action $*$ that is free and such that for all $t^n \in \mathbb{Z} = \langle t \rangle$, the set $\{g^{-1}(g * t^n) \mid g \in G\}$ is finite.

In particular, for every $i \in I$ there is a translation like action $*_i$ of $\mathbb{Z}$ on G_i. Now, define $K_i = \{g^{-1}(g *_i t) \mid g \in G_i\}$ and define $\rho_i|_{G_i}$ by

$$\forall g \in G_i, \rho_i(g) = (g^{-1}(g *_i t), g^{-1}(g *_i t^{-1})), \mathbf{b}).$$

Then, for all $n \in \mathbb{N}$, for all $g \in G_i$, we have $g\gamma^i_g(n, \rho) = g *_i t^n$. Since the action is free, the path is injective. Extend $\rho_i|_{G_i}$ by the trivial extension, that is, define $\rho_i(g) = \rho_i|_{G_i}(\pi_{A_i}(g))$ and now, for all $g \in \Gamma$, the paths $g\gamma^i_g(n, \rho) = g *_i t^n$ are also injective.

Now, it was proven in [5] that for every $j \in J$, there exists $K_j \Subset G_j$ such that the paradoxical subshift $\rho_j|_{G_j}$ on G_j is non-empty. Now define ρ_j as the trivial extension of any configuration of the paradoxical subshift on G_j, that is, define $\rho_j(g) = \rho_j|_{G_j}(\pi_{B_j}(g))$ for every $j \in J$. Since the map $(g, n) \in G_j \times \mathbb{N} \mapsto g\gamma^j_g(n + 1, \rho)$ is already injective, it follows that the path $n \in \mathbb{N} \mapsto \gamma^j_g(n, \rho)$ is injective.

It is straightforward to check that $\rho \in \mathcal{P}$. $\square$

The previous claim shows that it is possible to find infinite paths beginning at every point of the group in every direction. In the amenable directions, these paths are pairwise disjoint, but two distinct elements of $\Gamma(G)$ may have the same path. By contrast, in the non-amenable directions, two distinct elements g, h of $\Gamma(G)$ have disjoint paths (if we discount the elements g, h themselves). This is why two colors are necessary in the non-amenable directions - one color is used for the root of the path and another for the rest.

In general, the proof is adapted from the special case of RAAGs (whose outline was given in the beginning of the section) to general graph products in the following way. At each g, we choose a set of vertices $\mathbf{B}(g)$. We write $\mathcal{B}(g)$ the set of elements that can be reached by beginning at g and following a path in one of the directions of $\mathbf{B}(g)$. The point is that we want that if the intersection $\mathcal{B}(g) \cap \mathcal{B}(h)$ is non-empty and $\mathbf{B}(g) = \mathbf{B}(h)$, then $g = h$. If this is the case, then by having as many layers of computation as there are possible subgraphs $\mathbf{B}(g)$, g can use the entire set $\mathcal{B}(g)$ to do its computation

We have shown that we can choose disjoint paths at every point in the non-amenable directions, and so the non-amenable vertices may be used in $\mathbf{B}(h)$ for every h without overlap. In the case of an amenable vertex A_i, we will need to be more careful. The idea in this case is that we only include A_i in $\mathbf{B}(g)$ if h cannot be written in reduced form so that it ends with an element of A_i. Namely, in this case it happens that h is the *only* element of $\mathcal{B}(h)$ that cannot be written in reduced form so that it ends with an element of $\mathbf{B}(h) \cap \{A_i \mid i \in I\}$. We explain this in detail in Claim 3.2.

Let $\Gamma \curvearrowright X \subset \{0,1\}^\omega$ be an effectively closed action. We recall the notion of its set representation from [5]:

Definition 3. *If* $S \Subset \Gamma$ *is a finite generating set for the group* Γ *and* $\Gamma \curvearrowright X \subset \{0,1\}^\omega$ *is an action, the corresponding* set representation *of the action is*

$$\{x \in (\{0,1\}^S)^\omega \mid (x_n(1_\Gamma))_{n\in\mathbb{N}} \in X \text{ and } \forall s \in S, (x_n(s))_{n\in\mathbb{N}} = s(x_n(1_\Gamma))_{n\in\mathbb{N}}\}$$

Lemma 3.2 ([5], Remark 2.7)**.** *An action is effective if and only if its set representation is effectively closed for some generating set, in which case it is effectively closed for all generating sets.*

Let now $Y \subset \Omega^{\mathbb{N}}$ be the set representation of $\Gamma \curvearrowright X$ for the generating set S. Recall that $\Omega = \{0,1\}^S$ and $Y = \{y \in \Omega^{\mathbb{N}} \mid (y_n(1_\Gamma))_{n\in\mathbb{N}} \in X \text{ and } \forall s \in S, (y_n(s))_{n\in\mathbb{N}} = s(y_n(1_\Gamma))_{n\in\mathbb{N}}\}$.

Since $\Gamma \curvearrowright Y$ is effectively closed, there exists a Turing machine $\mathcal{M}$ that recognizes the forbidden patterns of Y. Alternatively, the computation of $\mathcal{M}$ on input $y \in (\Omega^S)^{\mathbb{N}}$ terminates if and only if $y \notin Y$.

A colored edge is an ordered pair $((u, c_u), (v, c_v))$ where $\{u, v\} \in E, c_u, c_v \in \{\mathbf{r}, \mathbf{c}, \mathbf{b}\}$. Denote by $\mathcal{E}$ the set of colored edges. A colored vertex is an element of $V \times \{\mathbf{r}, \mathbf{c}, \mathbf{b}\}$. Denote by $\mathcal{V}$ the set of colored vertices.

Any function defined on edges (respectively vertices) is extended trivially to colored edges (respectively colored vertices) by ignoring the color. Let also $\square$ be a blank symbol. Let $A = \Sigma \times 2^V \times 2^{\mathcal{E}} \times 2^{\mathcal{V}\times 2^V} \times (\Omega \cup \{\square\})^{E\times 2^V}$, and define the bush subshift $\mathcal{S}$ on alphabet A by demanding that the following conditions hold for every $\mathbf{S} = (\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}) \in \mathcal{S}$ at every $g \in \Gamma$.

1. $\rho \in \mathcal{P}$.
2. $\mathbf{B}(g)$ contains at least two nodes and the induced subgraph of (V, E) on these nodes is connected.
3. For every $v \in V$, for every $a \in S_v, \mathbf{B}(ga) \cap \mathbf{B}(g) \cap \mathrm{link}(v) \neq \emptyset$.
4. If $u, v \in \mathbf{B}(g)$ and $\{u, v\} \in E$, then $((u, \overline{c_u(g)}), (v, \overline{c_v(g)})) \in \mathbf{D}(g)$.
5. If $((u, c_u), (v, c_v)) \in \mathbf{D}(g)$, then $((u, c_u), (v, c_v)) \in \mathbf{D}(g\ell_u^{c_u}(g))$ and $((u, c_u), (v, c_v)) \in \mathbf{D}(g\ell_v^{c_v}(g))$.
6. If $u \in \mathbf{B}(g)$, then $((u, \overline{c_u(g)}), \mathbf{B}(g)) \in \mathbf{I}(g)$.
7. If $((u, c_u), C) \in \mathbf{I}(g)$ then $((u, c_u), C) \in \mathbf{I}(g\ell_u^{c_u}(g))$.
8. If $((u, c_u), (v, c_v)) \in \mathbf{D}(g)$, then for any $C \subset V$ such that $\{u, v\} \subset C, \mathbf{L}(g\ell_u^{c_u}(g))(\{u, v\}, C) = \mathbf{L}(g\ell_v^{c_v}(g))(\{u, v\}, C)$.
9. If $((u, c_u), C) \in \mathbf{I}(g)$ then for any $u' \in \mathrm{link}(u)$, for any $a \in S_{u'}$, for any $((u, c_u), C') \in \mathbf{I}(ga)$ and for any $\{u, v\} \subset C, \{u, v'\} \subset C'$, $\mathbf{L}(ga)(\{u, v'\}, C')(1_\Gamma) = \mathbf{L}(g)(\{u, v\}, C)(a^{-1})$.

The role of the bush subshift is the following. $\mathbf{B}$ chooses a bush, that is a set of directions at every vertex g, which is supposed to satisfy that if $g\langle\mathbf{B}(g)\rangle = h\langle\mathbf{B}(h)\rangle$ and $\mathbf{B}(g) = \mathbf{B}(h)$ then $g = h$. Hence, the layer of index $\mathbf{B}(g)$ of the bush subshift on the subset $g\langle\mathbf{B}(g)\rangle$ may be used entirely by g for computation without interference.

$\mathbf{D}(g)$ identifies the planes that are entirely contained in $\mathbf{B}(g)$ on which we will later be able to embed the Wang tiles that will do the actual computation (rule 2 ensures that there will be at least one plane on which to do computation).

$\mathbf{I}$ identifies the edges of $g\langle\mathbf{B}(g)\rangle$, that is the paths where it stays at bounded distance from another bush (if u commutes with some group G_v, then the paths $g\gamma_g^v(n, \rho)$ and $gu\gamma_{gu}^v(n, \rho)$ always stay close). By rule 3, we know that the bushes corresponding to two adjacent elements will always share an edge, so that synchronization does happen.

Finally, $\mathbf{L}$ contains the layer on which configurations of $\Gamma \curvearrowright X$ will be stored. By synchronizing $\mathbf{L}$ along diagonals of any plane contained in the bush (rule 8), we ensure that the same configuration of $\Gamma \curvearrowright X$ is written along any edge of $g\langle \mathbf{B}(g) \rangle$. Rule 9 ensures that the bushes that have two paths that stay close are synchronized.

For proving that $\mathcal{S}$ is non-empty, we will use some basic results about words in graph products.

A *writing* of an element $g \in \Gamma(G)$ is a sequence $g_1, \cdots, g_n$ such that each g_i belongs to some G_v and $g = g_1 \dots g_n$. The elements g_i of the sequence are called *syllables.*

Clearly, the element represented by the writing $g_1 \dots g_n$ is not modified by permuting g_i and g_{i+1} if $g_i \in G_u, g_{i+1} \in G_v$ and $\{u, v\} \in E$. Similarly, the element is not modified by replacing g_i, g_{i+1} by their product $g_i g_{i+1}$ if $g_i, g_{i+1} \in G_v$. Finally, the element is not modified by deleting g_i if $g_i = 1_{G_v}$.

We say that a writing $g = g_1 \dots g_n$ is *graphically reduced* if it cannot be shortened by applying the three operations above.

Definition 4. *If $g \in \Gamma(G)$, we define* tail(g) *as the set of vertices $v \in V$ such that there exists a graphically reduced writing of g ending with an element of G_v.*

Lemma 3.3. *The set* tail(g) *is a clique.*

Proof. Suppose $v, v' \in \text{tail}(g)$. Let w be a writing of g ending with an element of G_v, and w' one ending with an element of $G_{v'}$. Then one can turn w into w' by swapping adjacent commuting group elements in the writing [14]. In other words, there is a sequence of writings $w_0, w_1, \dots, w_k$ of g where for all i we can write $w_i = gabh, w_{i+1} = gbah$ where $a \in G_u, b \in G_{u'}$ with $(u, u') \in E$.

In particular, the rightmost syllable of w_0 corresponds some syllable in w_k, and it was moved there by pairwise swaps of elements coming from commuting groups. At some point, it thus had to swap with the rightmost syllable of w_k. This means $(v, v') \in E$. ☐

In the case of a RAAG, the clique tail(g) corresponds to the rightmost clique in the normal form described in [20].

Claim 3.2. *If G is atomic, then the $\Gamma(G)$-subshift $\mathcal{S}$ is a non-empty SFT.*

Proof. It is clear by the definition that $\mathcal{S}$ is of finite type. Let $\rho \in \mathcal{P}$ as in Claim 3.1, i.e. such that the paths along every direction are injective. Let $x \in X$, and for all $g \in \Gamma, y^{(g)} \in Y$ such that $(y_n^{(g)}(1_\Gamma))_{n \in \mathbb{N}} = g^{-1}x$.

Then define $\mathbf{B} \in 2^{V^\Gamma}$ by $\mathbf{B}(g) = (V \setminus \text{tail}(g)) \cup \{B_j | j \in J\}$.

Define $\mathbf{D}(g)$ by $\forall((u, c_u), (v, c_v)) \in \mathcal{E}, ((u, c_u), (v, c_v)) \in \mathbf{D}(g)$ if and only if $\exists g_0 \in \Gamma, n, m \in \mathbb{N}$ such that $\{u, v\} \subset \mathbf{B}(g_0), c_u = \overline{c_u(g_0)}, c_v = \overline{c_v(g_0)}$ and $g = g_0 \gamma_{g_0}^u(n, \rho) \gamma_{g_0 \gamma_{g_0}^u(n,\rho)}^v(m, \rho)$.

Define $\mathbf{I}$ by $\forall((u, c_u), C) \in \mathcal{V} \times 2^V, ((u, c_u), C) \in \mathbf{I}(g)$ if and only if $\exists g_0 \in \Gamma, n \in \mathbb{N}, c_u = \overline{c_u(g_0)}$ such that $C = \mathbf{B}(g)$ and $g = g_0 \gamma_{g_0}^v(n, \rho)$.

Define $\mathbf{L}$ by $\mathbf{L}(g \gamma_g^u(n, \rho) \gamma_{g \gamma_g^u(n,\rho)}^v(m, \rho))(\{u, v\}, \mathbf{B}(g)) = y_{n+m-1}^{(g)}$ for $\{u, v\} \in E, \{u, v\} \subset \mathbf{B}(g), n + m \geq 1$ and $\mathbf{L}(h)(\{u, v\}, C) = \square$ everywhere else.

Note that $\mathbf{L}$ is well defined, because if there exists $g\gamma_g^u(n,\rho)\gamma^v_{g\gamma_g^u(n,\rho)}(m,\rho) = g'\gamma_{g'}^{u'}(n',\rho)\gamma^v_{g'\gamma_{g'}^{u'}(n,\rho)}(m,\rho), \mathbf{B}(g) = \mathbf{B}(g')$ and $\{u,v\} = \{u',v'\}$, then since the bushes $g\langle \mathbf{B}(g) \cap \{A_i \mid i \in I\}\rangle$ and $g'\langle \mathbf{B}(g) \cap \{A_i \mid i \in I\}\rangle$ only have one point the last clique of which contains no element of $\mathbf{B}(g) \cap \{A_i \mid i \in I\}$ (respectively $\mathbf{B}(g') \cap \{A_i \mid i \in I\}$), it must be that G_u, G_v, G'_u and G'_v are non-amenable. But then the injectivity of $(n,g) \mapsto g\gamma_g^j(n,\rho)$ yields that $g = g'$ and it follows that $n = n', m = m'$, because the paths $n \in \mathbb{N} \mapsto \gamma_g^u(n,\rho)$ are injective by Claim 3.1.

We argue that $(\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}) \in \mathcal{S}$.

1. This is part of the definition.

2. $\mathbf{B}(g) = G \backslash (\mathrm{tail}(g) \cap \{G_i \mid i \in I\})$ is the complement of an amenable clique, so by Lemma 3.1, $\mathbf{B}(g)$ contains at least two nodes and is connected.

3. Let $g \in \Gamma, v \in V, a \in S_v$. There are two cases. If $v \in \mathbf{B}(g)$, then since by the previous point $\mathbf{B}(g)$ is connected and contains at least two nodes, there exists $u \in \mathbf{B}(g) \cap \mathrm{link}(v), u \neq v$. But then either G_u is non-amenable, and so $u \in \mathbf{B}(g) \cap \mathbf{B}(ga) \cap \mathrm{link}(v)$, or $u \notin \mathrm{tail}(g)$. In the latter case since $v \neq u$, it follows that $u \notin \mathrm{tail}(ga)$, and $u \in \mathbf{B}(g) \cap \mathbf{B}(ga) \cap \mathrm{link}(v)$.

   Otherwise if $v \notin \mathbf{B}(g)$, then by Lemma 3.1, there is $u \in \mathbf{B}(g) \cap \mathrm{link}(v)$. But $\mathrm{tail}(ga) \subset \mathrm{tail}(g) \cup \{v\}$, and therefore either $u \notin \mathrm{tail}(ga)$ or u is non-amenable, so that $u \in \mathbf{B}(ga)$.

4. This follows from the definition of $\mathbf{D}$ with $n = m = 0$.

5. This follows from the definition of $\mathbf{D}$ and the fact that $\ell_u^{c_u}(g)$ and $\ell_v^{c_v}(g_0)$ commute.

6. This follows from the definition of $\mathbf{I}$ with $n = 0$.

7. This follows from the definition of $\mathbf{I}$ with $g = g_0\gamma_g^v(n,\rho)$ and $h = g_0\gamma_{g_0}^v(n+1,\rho)$.

8. This follows from the definition of $\mathbf{L}$ and the fact that $\mathbf{D}(g)$ is empty on elements not of the form $g_0\gamma_{g_0}^u(n,\rho)\gamma^v_{g_0\gamma_{g_0}^u(n,\rho)}(m,\rho)$. Indeed, if $h = g_0\gamma_{g_0}^u(n,\rho)\gamma^v_{g_0\gamma_{g_0}^u(n,\rho)}(m,\rho)$ with $\{u,v\} \subset \mathbf{B}(g_0)$, then the condition is verified at $\mathbf{L}(h)(\{u,v\}, \mathbf{B}(g_0))$, and since $\mathbf{L}(h)(\{u,v\}, C) = \square$ whenever $C \neq \mathbf{B}(g_0)$, the condition is also verified in this case.

9. Assume $g = g_0\gamma_{g_0}^u(n,\rho)$ with $c_u = \overline{c_u(g_0)}$ and $C = \mathbf{B}(g_0)$. Then by definition of $\mathbf{L}$, we have $\mathbf{L}(g)(\{u,v\}, C)(a) = (a^{-1}g_0^{-1}x)_{n-1}$. Since v is adjacent to u, it follows from commutativity that $ga = g_0a\gamma_{g_0}^u(n,\rho) = g_0a\gamma_{g_0a}^u(n,\rho)$ by rule 6 of the path subshift. Finally, since v and u are adjacent, $c_u(g_0a) = c_u(g_0) = \overline{c_u}$.

   Hence, the definition of $\mathbf{L}$ yields $\mathbf{L}(g_0a)(\{u,v'\}, C')(1_\Gamma) = y_{n-1}^{(g_0a)}(1_\Gamma) = (a^{-1}g_0^{-1}x)_{n-1}$. $\square$

The previous claim shows that it is possible to stitch a bush on every point of Γ. These bushes will next be used for computation, to ensure that the configurations Ω^ω on the paths starting at each g are indeed in Y. Then, as we clarify below, item 8 and item 9 above already guarantee that every configuration encodes the orbit of a point in X.

For W a Wang tileset containing a specific symbol **seed** $\in W$, consider the map η which associates to any valid tiling of the quarter plane $\mathbb{Z}^2$ with symbol **seed** at $(0,0)$ the contents of the bottom row starting at $(0,0)$. That is, $\eta(\tau) = \tau|_{\{(n,0) \mid n \geq 1\}}$.

It is known that there exists a tileset W such that η surjects valid tilings of $\mathbb{Z}^2$ with **seed** at $(0,0)$ onto inputs on which $\mathcal{M}$ does not terminate [21]. In other words, if we write $Y = \{y \in (\{0,1\}^S)^\omega \mid (y_n(1_\Gamma))_{n\in\mathbb{N}} \in X$ and $\forall s \in S, (y_n(s))_{n\in\mathbb{N}} = s(y_n(1_\Gamma))_{n\in\mathbb{N}}\}$, then we have $W \supset \Omega$ and

$$\eta(\{\text{valid tilings } \tau : \mathbb{Z}^2 \to W \mid \tau(0,0) = \textbf{seed}\}) = Y.$$

Now define the computation subshift $\mathcal{Z}$ on alphabet $\Sigma \times 2^V \times 2^{\mathcal{E}} \times 2^{\mathcal{V}\times 2^V} \times (\Omega \cup \{\square\})^{E\times 2^V} \times 2^{\mathcal{E}} \times W^{\mathcal{E}}$ by demanding the following conditions of every $\mathbf{S} = (\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}, \mathbf{P}, \mathbf{T}) \in \mathcal{Z}$ at every $g \in \Gamma$.

1. $(\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}) \in \mathcal{S}$.
2. $\exists! e_g = \{u, v\} \subset E \cap \mathbf{B}(g), ((u, \overline{c_u(g)}), (v, \overline{c_v(g)})) \in \mathbf{P}(g)$.
3. If $e = ((u, c_u), (v, c_v)) \in \mathbf{P}(g)$ then $e \in \mathbf{P}(g\ell_u^{c_u}(g))$ and $e \in \mathbf{P}(g\ell_v^{c_v}(g))$.
4. $\mathbf{T}(g)(((u, \overline{c_u(g)}), (v, \overline{c_v(g)}))) = \textbf{seed}$.
5. If $e = ((u, c_u), (v, c_v)) \in \mathbf{P}(g)$ and $((u, c_u), C) \in \mathbf{I}(g)$ and $\mathbf{T}(g)(e) \neq \textbf{seed}$ then $\mathbf{L}(g)(\{u, v\}, C) = \mathbf{T}(g)(e)$.
6. If $e = ((u, c_u), (v, c_v)) \in \mathbf{P}(g)$ then $(\mathbf{T}(g)(e), \mathbf{T}(g\ell_u^{c_u}(g))(e), \mathbf{T}(g\ell_v^{c_v}(g))(e), \mathbf{T}(g\ell_u^{c_u}(g)^{-1})(e), \mathbf{T}(g\ell_v^{c_v}(g)^{-1})(e))$ is a valid pattern of W.

$\mathcal{Z}$ is clearly an SFT. Also define

$$\begin{aligned} \beta : \mathcal{Z} &\to \{0,1\}^{\mathbb{N}} \\ (\rho, \mathbf{B}, \llcorner\!\lrcorner, \llcorner\!\lrcorner, \mathbf{L}, \llcorner\!\lrcorner, \llcorner\!\lrcorner) &\mapsto (\mathbf{L}(\gamma_{1_\Gamma}^v(n+1, \rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} \text{ where } v \in \mathbf{B}(1_\Gamma). \end{aligned}$$

The following claim proves that the choice of $v \in \mathbf{B}(1_\Gamma)$ is inconsequential.

Claim 3.3. *For every $u, v \in \mathbf{B}(1_\Gamma)$, and for every $(\rho, \mathbf{B}, \llcorner\!\lrcorner, \llcorner\!\lrcorner, \mathbf{L}, \llcorner\!\lrcorner, \llcorner\!\lrcorner) \in \mathcal{Z}$, $(\mathbf{L}(\gamma_{1_\Gamma}^v(n+1, \rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} = (\mathbf{L}(\gamma_{1_\Gamma}^u(n+1, \rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}}$.*

Proof. By the second condition of the bush subshift, $\mathbf{B}(1_\Gamma)$ is connected, and so it suffices to show it for two adjacent vertices. Let us hence assume that $v \in \mathrm{link}(u)$.

If $v = u$, the result is trivial.

If not, by conditions 4 and 5 of the bush subshift, $\forall n, m \in \mathbb{N}, \{u, v\} \in \mathbf{D}(\gamma_{1_\Gamma}^u(n, \rho)\gamma_{\gamma_{1_\Gamma}^u(n,\rho)}^v(m, \rho))$. But then by condition 6, it is straightforward to show by induction that $\forall n \in \mathbb{N}, \forall k \leq n$, $\mathbf{L}(\gamma_{1_\Gamma}^u(n-k, \rho)\gamma_{\gamma_{1_\Gamma}^u(n-k,\rho)}^v(k, \rho))(\{u, v\}, \mathbf{B}(g)) = \mathbf{L}(\gamma_{1_\Gamma}^u(n, \rho))(\{u, v\}, \mathbf{B}(g))$, and the result follows from the case of $k = n$. $\square$

In the following, we will prove that $\Gamma \curvearrowright X$ is a topological factor of $\mathcal{Z}$ through β.

Claim 3.4. $\beta(\mathcal{Z}) \subset X$.

Proof. By condition 2 of the computation subshift, there exists
$e_{1_\Gamma} = ((u, \overline{c_u(g)}), (v, \overline{c_v(g)})) \in \mathbf{P}(1_\Gamma)$ such that $u, v \in \mathbf{B}(1_\Gamma)$. But then by condition 3, $\forall n, m \in \mathbb{N}, e_{1_\Gamma} \in \mathbf{P}(\gamma^u_{1_\Gamma}(n,\rho)\gamma^v_{\gamma^u_{1_\Gamma}(n,\rho)}(m,\rho))$. By condition 4, $\mathbf{T}(1_\Gamma)(e_{1_\Gamma}) = \textbf{seed}$ and by condition 6, $\mathbf{T}$ defines a valid tiling at every $n, m \in \mathbb{N}$. But any valid W-tiling of the quarter plane with **seed** at the origin must have an element of Y as a first row, and so $(\mathbf{T}(\gamma^u_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}))_{n\in\mathbb{N}} \in Y$.

Finally, note that $\forall n \in \mathbb{N}, ((u, \overline{c_u(g)}), \mathbf{B}(1_\Gamma)) \in \mathbf{I}(\gamma^u_{1_\Gamma}(n+1,\rho))$ and $\mathbf{T}(\gamma^u_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}) \neq \textbf{seed}$.

Hence, by the 5th condition,
$((\mathbf{L}(\gamma^u_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} = ((\mathbf{T}(\gamma^u_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma})(1_\Gamma))_{n\in\mathbb{N}} \in X$. ☐

Claim 3.5. *β is Γ-equivariant.*

Proof. Let $v \in V, s \in S_v$. Note that the case of $s = 1_\Gamma$ was proven in Claim 3.3. Otherwise, by condition 2 of the bush subshift, there exists $u \in \mathbf{B}(1_\Gamma) \cap \mathbf{B}(s) \cap \operatorname{link}(v)$. Then, conditions 7 and 8 ensure that $u \in \mathbf{I}(\gamma^v_{1_\Gamma}(n,\rho)) \forall n \in \mathbb{N}$. By condition 9 since $v \in \operatorname{link}(u)$,
$(\mathbf{L}(\gamma^u_{1_\Gamma}(n,\rho)s)(e, \mathbf{B}(s))(1_\Gamma))_{n\in\mathbb{N}} = (\mathbf{L}(\gamma^u_{1_\Gamma}(n,\rho))(f, \mathbf{B}(1_\Gamma))(s^{-1})))_{n\in\mathbb{N}}$ every time $v \in e \subset \mathbf{B}(1_\Gamma)$ and $v \in f \subset \mathbf{B}(s)$. Hence,

$$\begin{aligned}
s\beta((\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}, \mathbf{P}, \mathbf{T})) &= s(\mathbf{L}(\gamma^v_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} \\
&= (\mathbf{L}(\gamma^v_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(s))_{n\in\mathbb{N}} \text{ by Claim 3.4.} \\
&= (\mathbf{L}(\gamma^u_{1_\Gamma}(n+1,\rho))(e_{1_\Gamma}, \mathbf{B}(1_\Gamma))(s))_{n\in\mathbb{N}} \text{ by Claim 3.3.} \\
&= (\mathbf{L}(\gamma^u_{1_\Gamma}(n+1,\rho)s^{-1})(e_{s^{-1}}, \mathbf{B}(s^{-1}))(1_\Gamma))_{n\in\mathbb{N}} \\
&= (\mathbf{L}(s^{-1}\gamma^u_{1_\Gamma}(n+1,\rho))(e_{s^{-1}}, s\mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} \text{ as } s \text{ commutes with } G_u. \\
&= (s\mathbf{L})(\gamma^u_{1_\Gamma}(n+1,s\rho))(e_{s^{-1}}, s\mathbf{B}(1_\Gamma))(1_\Gamma))_{n\in\mathbb{N}} \\
&= \beta((s\rho, s\mathbf{B}, s\mathbf{D}, s\mathbf{I}, s\mathbf{L}, s\mathbf{P}, s\mathbf{T})).
\end{aligned}$$

This concludes the proof as S generates Γ. ☐

Claim 3.6. *If G is atomic, then $\beta : \mathcal{Z} \to X$ is surjective.*

Proof. Let $x \in X$, and for all $g \in \Gamma, y^{(g)} \in Y$ such that $(y^{(g)}_n(1_\Gamma))_{n\in\mathbb{N}} = g^{-1}x$. Define $\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}$ and $\mathbf{L}$ as in Claim 3.2.

Now, for every $g \in \Gamma$, choose one edge $\{u_g, v_g\} \subset \mathbf{B}(g)$, and set $e_g = ((a_g, \overline{c_{a_g}(g)}), (v, \overline{c_{b_g}(g)}))$.

Define $\mathbf{P}$ by $\forall e = ((u, c_u), (v, c_v)) \in \mathcal{E}, e \in \mathbf{P}(g)$ if and only if $\exists n, m \in \mathbb{N}, \exists h \in \Gamma$ such that $g = h\gamma^u_h(n,\rho)\gamma^v_{h\gamma^u_h(n,\rho)}(m,\rho)$ and $e_h = e$.

Also define $\mathbf{T}(g)(e_g) = \textbf{seed}$ and $\forall n \in \mathbb{N}, \mathbf{T}(g\gamma^{u_g}_g(n+1,\rho))(e_g) = y^{(g)}_n$. Extend $\mathbf{T}$ so that $\mathbf{T}(g\gamma^{u_g}_g(n,\rho)\gamma^{v_g}_{g\gamma^{u_g}_g(n,\rho)}(m,\rho))(e_g)$ is defined by using the compatibility conditions of W. This definition is justified by the same remark as the definition of $\mathbf{L}$ in Claim 3.2. Now extend $\mathbf{T}$ to every layer on every point to satisfy condition 5.

Note that this condition only applies to elements of the form $g_0\gamma^u_{g_0}(n,\rho)$ for some $n > 1$. Finally, extend arbitrarily $\mathbf{T}$ to every layer on every point.

We claim that $(\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}, \mathbf{P}, \mathbf{T}) \in \mathcal{Z}$.

1. This follows from Claim 3.2.
2. This follows from the definition of $\mathbf{P}$ with $h = g, n = m = 0$.
3. This follows from the definition of $\mathbf{P}$ with $g = g_0\gamma^u_{g_0}(n,\rho)\gamma^v_{g_0\gamma^u_{g_0}(n,\rho)}(m,\rho)$, as then $g\ell_u^{c_u}(g) = g_0\gamma^u_{g_0}(n+1,\rho)\gamma^v_{g_0\gamma^u_{g_0}(n,\rho)}(m,\rho)$, and $g\ell_b^{c_v}(g) = g_0\gamma^u_{g_0}(n,\rho)\gamma^v_{g_0\gamma^u_{g_0}(n,\rho)}(m+1,\rho)$.
4. This is part of the definition of $\mathbf{T}$.
5. This follows from the definition of $\mathbf{P}, \mathbf{L}$ and $\mathbf{T}$ when $g = g_0\gamma^u_{g_0}(n,\rho)$, as then $\mathbf{L}(g)(\{u,v\})(C) = \mathbf{L}(g)(\{u,v\})(\mathbf{B}(g_0)) = y^{(g)}_{n-1} = \mathbf{T}(g)(e_{g_0}) = \mathbf{T}(g)(e)$. But $\mathbf{I}$ is empty outside of these, and so the condition is verified everywhere.
6. This is part of the definition on every $g_0\gamma^{u_{g_0}}_{g_0}(n,\rho)\gamma^{v_{g_0}}_{g_0\gamma^{u_g}_{g_0}(n,\rho)}(m,\rho)$. But $\mathbf{P}$ is empty outside of these, and so the condition is verified everywhere.

But finally, from the definition of $\mathbf{L}$ it is clear that $\beta((\rho, \mathbf{B}, \mathbf{D}, \mathbf{I}, \mathbf{L}, \mathbf{P}, \mathbf{T})) = x$, and so β is surjective. ☐

Theorem 1 is now clear.

Proof of Theorem 1. Let G be a graph that is not a clique or contains at least two non-amenable vertices.

If G has no disconnecting amenable clique, then it is atomic and so for any effectively closed action $\Gamma(G) \curvearrowright X \subset \{0,1\}^\omega$, Claims 3.4, 3.5 and 3.6 show that $\Gamma \curvearrowright X$ is a topological factor of the subshift of finite type $\mathcal{Z}$ through β.

Conversely, if G has a disconnecting amenable clique, then by Corollary 1.1.1, it is not self-simulable. ☐

We then obtain Theorem 2 as a corollary of Theorem 1.

Proof of Theorem 2. This is an immediate consequence of Theorem 1 in the case where every vertex is infinite and amenable. ☐

4 Questions

Our self-simulability construction requires the node groups G_u to be infinite. In fact, bushes may not be generalized easily to the case of finite groups. Indeed, the graph product of a square of $\mathbb{Z}_3$s is self-simulable because it is a direct product of two non-amenable groups. However, the complement of an edge is another edge, which is finite and hence cannot be used as a bush.

Question 2. *If we allow finite node groups, when is a graph product self-simulable?*

In particular, we can ask the following if every node group is $\mathbb{Z}_2$.

Question 3. *Which right-angled Coxeter groups are self-simulable?*

The characterization is not the same as with RAAGs. Indeed, as we showed in Example 1, a cycle of length at least 4 always defines a self-simulable RAAG. However, for large enough cycles, any two disconnected nodes give a copy of the amenable group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ that disconnects the group. On the other hand, we do not know if the triangular prism graph of Figure 4 generates a self-simulable right-angled Coxeter group. Indeed, it has no disconnecting amenable subgraph, but the complement of the red clique generates a finite group, so that the bush method cannot work as is.

If a right-angled Coxeter group is quasi-isometric to a right-angled Artin group, then because both are finitely-presented, [5, Theorem 1.5] implies that their self-simulability statuses are the same. It seems to be unknown which right-angled Coxeter groups have this property, but partial results are given in [10].

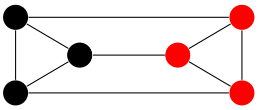

Figure 4: Prism graph.

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