

## ON SURJUNCTIVE AND INJUNCTIVE SUBSHIFTS OF FINITE TYPE

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ABSTRACT. A dynamical system is said to be surjunctive if every injective endomorphism of the system is surjective and it is said to be injunctive if every surjective endomorphism is injective. An endomorphism of a dynamical system is called pre-injective if its restriction to every homoclinicity class of the phase space is injective. One says that a dynamical system has the Moore property if every surjective endomorphism of the system is pre-injective and that it has the Myhill property if every pre-injective endomorphism is surjective. We give characterisations of surjunctivity and injunctivity for  $\mathbb{Z}$ -subshifts of finite type in terms of their irreducible components and their Cantor-Bendixson decomposition. We also prove that a  $\mathbb{Z}$ -subshift of finite type is surjunctive if and only if it has the Moore property and that every injunctive  $\mathbb{Z}$ -subshift of finite type is surjunctive. This implies in particular that a  $\mathbb{Z}$ -subshift of finite type has the Moore property whenever it has the Myhill property.

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## 1. INTRODUCTION

A self-map of a finite set is injective if and only if it is surjective. In a concrete category, one says that an object is *surjunctive* if every injective endomorphism is surjective [6], and *injunctive* if every surjective endomorphism is injective. These are natural finiteness properties of objects, and besides the category of finite sets, they hold in several other settings. For instance, in the category of vector spaces over a field, with linear maps as morphisms, every finite-dimensional object is surjunctive and injunctive. The Ax-Grothendieck theorem [2] implies surjunctivity for affine varieties over an algebraically closed field, with polynomials as morphisms. Surjunctivity plays an interesting role in the theory of cellular automata. It is known that for a large class of groups  $G$ , called *sofic groups*, for any finite alphabet  $A$ , all injective cellular automata  $f: A^G \rightarrow A^G$  (continuous maps commuting with the shift action of  $G$ ) are surjective [13], [24]. This can be stated by saying that full shifts over sofic groups are surjunctive in the category of subshifts. This result has been generalized in several ways. In particular, it is known that non-wandering Smale systems are surjunctive [9].

In the present paper, we fully characterize surjunctivity and injunctivity in the category of subshifts of finite type over the group  $\mathbb{Z}$  in terms of the poset structure of their irreducible components and their Cantor-Bendixson decomposition. It turns out that the surjunctive objects in this category are exactly those which satisfy the Moore property (which is a certain weakening of the injunctivity property). In addition, the Myhill property (which is a certain strengthening of the surjunctivity property) implies the Moore property.

We now proceed to more precise definitions and statements. Let  $A$  be a finite set. Consider the set  $A^{\mathbb{Z}}$  consisting of all sequences  $x = (x_i)_{i \in \mathbb{Z}}$  such that  $x_i \in A$  for all  $i \in \mathbb{Z}$ . Equip  $A^{\mathbb{Z}}$  with its *prodiscrete topology*, i.e., the product topology obtained by taking the discrete topology on every factor  $A$  of  $A^{\mathbb{Z}} = \prod_{\mathbb{Z}} A$ . The *shift map* on  $A^{\mathbb{Z}}$  is the homeomorphism  $\sigma: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined by  $(x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}$ . A closed  $\sigma$ -invariant subset  $X \subset A^{\mathbb{Z}}$  is called a *subshift*. A subshift  $X \subset A^{\mathbb{Z}}$  is said to be of *finite type* if there exist an integer  $n \geq 0$  and a subset  $S \subset A^n$  such that  $X$  consists of all  $x \in A^{\mathbb{Z}}$  that satisfy  $(x_{i+1}, x_{i+2}, \dots, x_{i+n}) \in S$  for all  $i \in \mathbb{Z}$ .

Let  $X \subset A^{\mathbb{Z}}$  be a subshift. An *endomorphism* of  $X$  is a continuous map  $\tau: X \rightarrow X$  which commutes with the shift map  $\sigma$ . These morphisms are also called *sliding-block codes* [18] or *one-dimensional cellular automata* [7, 8].

One says that the subshift  $X$  is *surjunctive* [12] if every injective endomorphism of  $X$  is surjective and that it is *injunctive*, or *coalescent* (cf. [1], [10]), if every surjective endomorphism of  $X$  is injective. Two sequences  $x, y \in X$  are called *homoclinic* (or *almost equal*) if the set of  $i \in \mathbb{Z}$  such that  $x_i \neq y_i$  is finite. Homoclinicity is an equivalence relation on  $X$ . One says that an endomorphism of  $X$  is *pre-injective* [13] if its restriction to every homoclinicity class of  $X$  is injective. Injectivity trivially implies pre-injectivity but the converse implication does not hold in general. One says that  $X$  has the *Moore property*, or that  $X$  is *Moore* [20], if every surjective endomorphism of  $X$  is pre-injective. One says that  $X$  has the *Myhill property*, or that  $X$  is *Myhill* [21], if every pre-injective endomorphism of  $X$  is surjective. Finally, one says that  $X$  satisfies the *Garden of Eden theorem* if  $X$  is both Moore and Myhill.

Subshifts over  $\mathbb{Z}$  form a concrete category where the morphisms are the continuous shift-commuting maps. We note that in this category, surjectivity and injectivity correspond exactly to the abstract properties of epicness and monicness (see [22, Table 1], where  $K2$  is the category of  $\mathbb{Z}$ -subshifts of finite type), so surjunctivity (resp., injunctivity) can equivalently be stated as the property that monic endomorphisms are epic (resp., epic endomorphisms are monic).

In the present paper, we show that surjunctivity is equivalent to the Moore property for subshifts of finite type. We prove it by showing that each of these properties is equivalent to the same condition on the poset structure of the irreducible components of the subshift (see Section 3 for precise definitions). Actually, we shall establish the following.

**Theorem 1.1.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then the following conditions are equivalent:*

- (a)  $X$  is surjunctive;
- (b)  $X$  is Moore;
- (c) every irreducible component of  $X$  is extremal and every infinite irreducible component of  $X$  is isolated;
- (d)  $X$  has Cantor-Bendixson rank at most 2 and the perfect kernel of  $X$  is the union of the isolated infinite irreducible components of  $X$ ;
- (e)  $X$  is the disjoint union of a non-wandering subshift of finite type and of a countable subshift of finite type having Cantor-Bendixson rank at most 2.

Since injectivity implies pre-injectivity, every subshift having the Myhill property is surjunctive. Thus, as an immediate consequence of Theorem 1.1, we get the following.

**Corollary 1.2.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. If  $X$  has the Myhill property then it has the Moore property. In other words, if  $X$  has the Myhill property then it satisfies the Garden of Eden theorem.*

The hypothesis that the subshift  $X$  is of finite type cannot be removed neither from Theorem 1.1 nor from Corollary 1.2. Indeed, Fiorenzi proved in [11] that the even subshift (i.e., the subshift consisting of all sequences of 0s and 1s satisfying the condition that there is always an even number of 0s between two 1s) is Myhill but not Moore. Note that the

even subshift is a sofic subshift (i.e., a factor of a subshift of finite type) and topologically mixing.

We shall also establish the following result.

**Theorem 1.3.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then the following conditions are equivalent:*

- (a)  $X$  is injunctive;
- (b) every irreducible component of  $X$  is finite and extremal;
- (c)  $X$  is countable with Cantor-Bendixson rank at most 2.

From Theorem 1.1 and Theorem 1.3, we deduce that a subshift of finite type is surjunctive if and only if it is the disjoint union of a non-wandering subshift of finite type and of an injunctive subshift of finite type.

Condition (c) in Theorem 1.3 is purely topological. We deduce that being injunctive is a topological invariant for subshifts of finite type. Thus, we have the following.

**Corollary 1.4.** *Let  $A$  and  $B$  be finite sets and let  $X \subset A^{\mathbb{Z}}$  and  $Y \subset B^{\mathbb{Z}}$  be subshifts of finite type. Suppose that the topological spaces underlying  $X$  and  $Y$  are homeomorphic. Then  $X$  is injunctive if and only if  $Y$  is injunctive.*

It follows from our results that subshifts of finite type satisfy the following:

$$\begin{array}{ccccc} & & \text{injunctive} & & \\ & & \downarrow & & \\ \text{Myhill} & \implies & \text{surjunctive} & \iff & \text{Moore} \end{array}$$

We shall give examples showing that each of the two one-sided implications appearing in the diagram above is strict (see Example 8.3 and Example 8.5). There exist even surjunctive (and hence Moore) subshifts of finite type that are neither Myhill nor injunctive (see Example 8.6).

The paper is organized as follows. In Section 2, we introduce the necessary terminology and collect some background material on directed graphs, dynamical systems, symbolic dynamics, and Cantor-Bendixson theory. Section 3 describes the poset  $\mathcal{C}$  of irreducible components of a subshift of finite type  $X$ . We prove that an irreducible component of a subshift of finite type  $X$  is isolated in the poset of all irreducible components of  $X$  if and only if it is open in  $X$ . We show that every endomorphism  $\tau$  of  $X$  naturally induces a poset endomorphism  $\rho_\tau$  of  $\mathcal{C}$  and that  $\rho_\tau$  is a poset automorphism provided that  $\tau$  is surjective or injective. We also give a characterization of isolated configurations in terms of their initial and terminal irreducible components. In Section 4, we use the technique of markers to construct endomorphisms of subshifts of finite type with an infinite irreducible component that are surjective but not injective. In Section 5, given an irreducible component  $C$  of a vertex subshift  $X$ , we construct an injective endomorphism  $\tau_C: X \rightarrow X$ . The fact that it may fail to be surjective is used in Section 7.2 for proving the properties of surjunctive subshifts of finite type stated in Theorem 1.1. In Section 6, given an infinite irreducible component  $C$  of a vertex subshift  $X$  which is minimal in the poset of all irreducible components of  $X$ , we construct a surjective endomorphism  $\tau^C: X \rightarrow X$ . The fact that it may

fail to be pre-injective is used in Section 7.2 for proving the properties of Moore subshifts of finite type stated in Theorem 1.1. The proofs of Theorem 1.1 and Theorem 1.3 are given in Section 7. Section 8 contains a series of examples. Subshifts of finite type are divided into five classes according to whether or not they satisfy surjectivity, injectivity, Moore, and Myhill. We give explicit examples for each of these classes. We also show that surjectivity (resp. Moore, resp. Myhill) is not a topological invariant for subshifts of finite type and that injectivity is not a topological invariant for sofic subshifts. In the final section, we have collected some questions that remain open.

## 2. PRELIMINARIES

In this section, we fix notation and collect the necessary material regarding posets, words, directed graphs, dynamical systems, symbolic dynamics, and Cantor-Bendixson theory. Some proofs are given. More details can be found in [4], [7], [8], [15], [17], [18], [23].

**2.1. General terminology.** We write  $\mathbb{Z}$  for the set of integers and  $\mathbb{N}$  for the set of non-negative integers (thus,  $0 \in \mathbb{N}$ ).

Given a set  $X$ , we write  $|X|$  for the cardinality of  $X$ . We denote by  $\text{Id}_X$  the identity map on  $X$ , i.e., the map  $\text{Id}_X: X \rightarrow X$  given by  $\text{Id}_X(x) := x$  for all  $x \in X$ . A *partition* of  $X$  is a set of pairwise disjoint non-empty subsets of  $X$  whose union is  $X$ .

Given a map  $f: X \rightarrow Y$  between sets and subsets  $X' \subset X$ ,  $Y' \subset Y$ , we define the subsets  $f(X') \subset Y$  and  $f^{-1}(Y') \subset X$  by  $f(X') := \{f(x) : x \in X'\}$  and  $f^{-1}(Y') := \{x \in X : f(x) \in Y'\}$ .

Given maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we denote by  $g \circ f$  the *composite* of  $f$  and  $g$ , i.e., the map  $g \circ f: X \rightarrow Z$  defined by  $(g \circ f)(x) := g(f(x))$  for all  $x \in X$ .

Given a map  $f: X \rightarrow X$ , we set  $f^0 := \text{Id}_X$  and define inductively  $f^n$  for  $n \in \mathbb{N}$  by setting  $f^{n+1} := f \circ f^n$ . If  $f$  is bijective, we write  $f^{-1}$  for the inverse of  $f$  and  $f^{-n} := (f^{-1})^n$  for every  $n \in \mathbb{N}$ .

**2.2. Posets.** Let  $X$  be a set and let  $\preceq$  be a binary relation on  $X$ . One says that  $\preceq$  is *reflexive* if one has  $x \preceq x$  for all  $x \in X$ . One says that  $\preceq$  is *antisymmetric* if  $x \preceq y$  and  $y \preceq x$  implies  $x = y$  for all  $x, y \in X$ . One says that  $\preceq$  is *transitive* if  $x \preceq y$  and  $y \preceq z$  imply  $x \preceq z$  for all  $x, y, z \in X$ . One says that  $\preceq$  is a *partial ordering* on  $X$  if  $\preceq$  is reflexive, antisymmetric, and transitive. If  $\preceq$  is a partial ordering on a set  $X$ , one says that  $(X, \preceq)$  is a *partially ordered set* or a *poset*.

Let  $(X, \preceq)$  be a poset. We write  $x \prec y$  to mean  $x \preceq y$  and  $x \neq y$ . One says that an element  $x \in X$  is *minimal* (resp. *maximal*) if there exists no  $y \in X$  such that  $y \prec x$  (resp.  $x \prec y$ ). One says that  $x \in X$  is *extremal* if  $x$  is minimal or maximal. One says that  $x \in X$  is *isolated* if  $x$  is both minimal and maximal. One says that  $x \in X$  is an *intermediate* element if there exist  $y, z \in X$  such that  $y \prec x \prec z$ . Observe that  $x$  is intermediate if and only if  $x$  is not extremal. One says that  $x, y \in X$  are *adjacent* if either  $x \prec y$  and there is no  $z \in X$  such that  $x \prec z \prec y$ , or  $y \prec x$  and there is no  $z \in X$  such that  $y \prec z \prec x$ . One says that a subset  $\Gamma \subset X$  is a *chain* if the restriction of  $\preceq$  to  $\Gamma$  is a total ordering, i.e., for all  $x, y \in \Gamma$ , one has  $x \preceq y$  or  $y \preceq x$ .

If  $(X, \preceq)$  and  $(X', \preceq')$  are posets, one says that a map  $\varphi: X \rightarrow X'$  is *order-preserving* if one has  $\varphi(x) \preceq' \varphi(y)$  for all  $x, y \in X$  such that  $x \preceq y$ . The *category of posets* is the category whose objects are posets and whose morphisms are order-preserving maps between them. We shall denote by  $\text{End}(X, \preceq)$  the endomorphism monoid of a poset  $(X, \preceq)$ .

**2.3. Words.** Let  $A$  be a set, called the *alphabet*, whose elements are called the *letters*. Denote by  $A^*$  the free monoid based on  $A$ . Every element  $w \in A^*$  can be uniquely written as a word  $w = a_1 a_2 \cdots a_n$ , where  $n \geq 0$  and  $a_i \in A$  for all  $i \in \{1, \dots, n\}$ . We then write  $|w| := n$ . Denoting by  $A^n$  the set of words  $w \in A^*$  such that  $|w| = n$ , we thus have  $A^* = \bigcup_{n \geq 0} A^n$ . The monoid operation on  $A^*$  is the concatenation of words. Thus, if  $w := a_1 \cdots a_n \in A^n$  and  $w' := a'_1 \cdots a'_m \in A^m$ , then  $ww' = a_1 \cdots a_n a'_1 \cdots a'_m \in A^{n+m}$ . The identity element of  $A^*$  is the *empty word*, that is, the unique element of  $A^0$ . Given a word  $w = a_1 \cdots a_n \in A^n$ ,  $n \geq 1$ , and an integer  $k \in \{1, \dots, n\}$ , the *k-prefix* (resp. *k-suffix*) of  $w$  is the word  $\text{pref}_k(w) := a_1 \cdots a_k \in A^k$  (resp.  $\text{suff}_k(w) := a_{n-k+1} \cdots a_n \in A^k$ ). The element  $\text{pref}_1(w) = a_1 \in A$  (resp.  $\text{suff}_1(w) = a_n \in A$ ) is the *initial* (resp. *terminal*) letter of  $w$ .

A *right-infinite* (resp. *left-infinite*, resp. *bi-infinite*) word over  $A$  is an element of  $A^{\mathbb{N}}$  (resp.  $A^{-\mathbb{N}}$ , resp.  $A^{\mathbb{Z}}$ ), that is, a sequence  $w = (a_n)_{n \in \mathbb{N}}$  (resp.  $w = (a_n)_{n \in -\mathbb{N}}$ , resp.  $w = (a_n)_{n \in \mathbb{Z}}$ ), where  $a_n \in A$  for all  $n \in \mathbb{N}$  (resp.  $n \in -\mathbb{N}$ , resp.  $n \in \mathbb{Z}$ ). We shall also use the notation  $w = a_0 a_1 a_2 \cdots$  (resp.  $w = \cdots a_{-2} a_{-1} a_0$ , resp.  $w = \cdots a_{-1} a_0 a_1 \cdots$ ) to denote  $w$ .

**2.4. Directed graphs.** A *directed graph* is a pair  $G = (V, E)$ , where  $V$  is a set and  $E$  is a subset of  $V^2$ . The set  $V$  is called the set of *vertices* of  $G$  and  $E$  is called its set of *edges*. A directed graph is said to be *finite* if its set of vertices is finite.

Let  $G = (V, E)$  be a directed graph.

Given a subset  $W \subset V$ , the pair  $G_W = (W, F)$ , where  $F := E \cap W^2$ , is also a directed graph. One says that  $G_W$  is the *subgraph induced by W*.

Given vertices  $v, v' \in V$  and an integer  $n \geq 1$ , a *path of length n* from  $v$  to  $v'$  in  $G$  is a word  $\pi = v_0 v_1 \cdots v_n \in V^{n+1}$  such that  $v_0 = v$ ,  $v_n = v'$ , and  $v_i v_{i+1} \in E$  for all  $i \in \{0, 1, \dots, n-1\}$ . Note that the length of  $\pi$  is equal to  $|\pi| - 1$  and that, by definition, the length of a path is always positive.

Let  $\pi = v_0 v_1 \cdots v_m \in V^{m+1}$  be a path in  $G$  of length  $m$  from a vertex  $v_0$  to a vertex  $v_m$  and let  $\pi' = v_m v_{m+1} \cdots v_{m+n} \in V^{n+1}$  be a path in  $G$  of length  $n$  from  $v_m$  to a vertex  $v_{m+n}$ . The *composite* of  $\pi$  and  $\pi'$  is the path in  $G$  of length  $m+n$  from  $v_0$  to  $v_{m+n}$  defined by  $\pi \# \pi' := v_0 v_1 \cdots v_m v_{m+1} \cdots v_{m+n} \in V^{m+n+1}$ . We define a relation  $\preceq$  on  $V$  by writing  $v \preceq v'$  if there exists a path from  $v$  to  $v'$  in  $G$ . Observe that  $\preceq$  is transitive on  $V$ . Indeed, if  $\pi$  is a path from  $v$  to  $v'$  and  $\pi'$  is a path from  $v'$  to  $v''$ , then  $\pi \# \pi'$  is a path from  $v$  to  $v''$ . A path from a vertex  $v$  to itself is called a *cycle* based at  $v$ . A cycle of length 1 is called a *self-loop*. Note that if  $\pi = v_0 v_1 \cdots v_{n-1} v_0$  is a cycle based at  $v_0$  of length  $n$  and  $k \geq 1$  is an integer, then  $\pi^{k\#} = (v_0 v_1 \cdots v_{n-1})^k v_0$  is a cycle based at  $v_0$  of length  $kn$ . One says that a cycle  $v_0 v_1 \cdots v_{n-1} v_0$  based at a vertex  $v_0$  is *simple* if the vertices  $v_0, v_1, \dots, v_{n-1}$  are all distinct.

A vertex  $v \in V$  is called *recurrent* if there exists a cycle (of positive length) based at  $v$ . A vertex which is not recurrent is said to be *transient*. We denote by  $R$  (resp.  $T$ ) the

set of all recurrent (resp. transient) vertices of  $G$ . Note that the restriction of  $\preceq$  to  $R$  is reflexive. We introduce a relation  $\sim$  on  $R$  by writing  $v \sim v'$  if there exist a path from  $v$  to  $v'$  and a path from  $v'$  to  $v$  (i.e., if  $v \preceq v'$  and  $v' \preceq v$ ). It is clear that  $\sim$  is an equivalence relation on  $R$ . The elements of the quotient set  $\mathcal{K} := R/\sim$ , i.e., the equivalence classes of  $\sim$  in  $R$ , are called the *path-components* of  $G$ . The relation  $\preceq$  naturally induces a relation, also denoted by abuse  $\preceq$ , on  $\mathcal{K}$ . Given  $K, K' \in \mathcal{K}$ , we have  $K \preceq K'$  if and only if  $v \preceq v'$  for some (or, equivalently, all)  $v \in K$  and  $v' \in K'$ . Clearly,  $\preceq$  is reflexive, antisymmetric, and transitive on  $\mathcal{K}$ . In other words,  $(\mathcal{K}, \preceq)$  is a poset. One says that the graph  $G$  is *strongly connected* if  $V \neq \emptyset$  and, for all  $v, v' \in V$ , there exists a path in  $G$  from  $v$  to  $v'$ . Thus,  $G$  is strongly connected if and only if  $\mathcal{K} = \{V\}$ .

**2.5. Dynamical systems.** A *dynamical system* is a pair  $(X, f)$ , where  $X$  is a compact metrizable space and  $f: X \rightarrow X$  is a homeomorphism. The space  $X$  is called the *phase space* of the dynamical system. When the homeomorphism  $f$  is clear from the context, we shall sometimes simply write  $X$  to designate the dynamical system  $(X, f)$ .

Let  $(X, f)$  be a dynamical system. The *orbit* of a point  $x \in X$  is the subset  $\mathcal{O}(x) \subset X$  given by  $\mathcal{O}(x) := \{f^n(x) : n \in \mathbb{Z}\}$ . A subset  $Y \subset X$  is said to be *f-invariant*, or simply *invariant*, if  $f(Y) = Y$ . If  $Y \subset X$  is an invariant subset, we denote by  $f|_Y: Y \rightarrow Y$  the restriction of  $f$  to  $Y$ . When  $Y \subset X$  is a closed invariant subset, the pair  $(Y, f|_Y)$  is also a dynamical system and one says that  $(Y, f|_Y)$  is a *subsystem* of  $(X, f)$ . We shall write such a subsystem  $(Y, f)$ , or even simply  $Y$ , if there is no risk of confusion.

A point  $x \in X$  is said to be *periodic* if there exists an integer  $n \geq 1$  such that  $f^n(x) = x$ . Such an integer  $n$  is then called a *period* of  $x$ . The point  $x$  is periodic if and only if  $\mathcal{O}(x)$  is finite. If  $x \in X$  is a periodic point, the least integer  $n \geq 1$  such that  $f^n(x) = x$  is called the *minimal period* of  $x$ . Observe that if  $x \in X$  is periodic with minimal period  $n$  then  $\mathcal{O}(x) = \{x, f(x), \dots, f^{n-1}(x)\}$  is finite with cardinality  $n$ . We shall denote by  $\text{Per}(X)$  the set of all periodic points of  $(X, f)$ .

A point  $x \in X$  is said to be *non-wandering* if for every neighborhood  $U$  of  $x$ , there exists an integer  $n \geq 1$  such that  $U \cap f^n(U) \neq \emptyset$ . The set  $\text{NW}(X)$  of all non-wandering points of  $f$  is a closed invariant subset of  $X$ . Observe that  $\text{Per}(X) \subset \text{NW}(X)$ .

One says that the dynamical system  $(X, f)$  is *non-wandering* if every point of  $X$  is non-wandering, i.e.,  $\text{NW}(X) = X$ . One says that  $(X, f)$  is *topologically mixing* if  $X$  is non-empty and, for all non-empty open subsets  $U, V \subset X$ , there exists an integer  $N \geq 1$  such that  $U \cap f^n(V) \neq \emptyset$  for all  $n \geq N$ . One says that  $(X, f)$  is *irreducible*, or *one-sided topologically transitive*, if  $X$  is non-empty and, for all non-empty open subsets  $U, V \subset X$ , there exists  $n \geq 1$  such that  $U \cap f^n(V) \neq \emptyset$ . The dynamical system  $(X, f)$  is irreducible if and only if there exists a point  $x \in X$  whose forward orbit  $\mathcal{O}^+(x) := \{f^n(x) : n \geq 1\}$  is dense in  $X$ . One says that  $(X, f)$  is *topologically transitive* if  $X$  is non-empty and, for all non-empty open subsets  $U, V \subset X$ , there exists  $n \in \mathbb{Z}$  such that  $U \cap f^n(V) \neq \emptyset$ . The dynamical system  $(X, f)$  is topologically transitive if and only if there exists a point  $x \in X$

whose orbit  $\mathcal{O}(x)$  is dense in  $X$ . We have the following implications:

$$\begin{array}{ccccc} \text{topologically mixing} & \implies & \text{irreducible} & \implies & \text{non-wandering} \\ & & \downarrow & & \\ & & \text{topologically transitive} & & \end{array}$$

Each of these implications is strict. It can be shown that a dynamical system is irreducible if and only if it is non-wandering and topologically transitive [23, Theorem 5.10]. Moreover, if  $X$  has no isolated points, then  $(X, f)$  is irreducible if and only if it is topologically transitive [4, Proposition 2.2.2].

Let  $(X, f)$  be a dynamical system. Given  $x \in X$ , the  $\alpha$ -limit set (resp.  $\omega$ -limit set) of  $x$  is the set  $\alpha(x) = \alpha_f(x)$  (resp.  $\omega(x) = \omega_f(x)$ ) consisting of all cluster points of the set  $\{f^{-n}(x) : n \in \mathbb{N}\}$  (resp.  $\{f^n(x) : n \in \mathbb{N}\}$ ) in  $X$ . Thus, a point  $y \in X$  is in  $\alpha(x)$  (resp.  $\omega(x)$ ) if and only if there exists a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that the sequence  $(f^{-n_k}(x))_{k \in \mathbb{N}}$  (resp.  $(f^{n_k}(x))_{k \in \mathbb{N}}$ ) converges to  $y$ . The sets  $\alpha(x)$  and  $\omega(x)$  are non-empty closed invariant subsets of  $\text{NW}(X)$  and one has

$$(2.1) \quad \alpha_f(x) = \omega_{f^{-1}}(x) \text{ and } \omega_f(x) = \alpha_{f^{-1}}(x)$$

for all  $x \in X$ .

**Proposition 2.1.** *Let  $(X, f)$  be a dynamical system and let  $x \in \text{Per}(X)$ . Then  $\alpha(x) = \omega(x) = \mathcal{O}(x)$ . In particular,  $\alpha(x)$  and  $\omega(x)$  are finite sets whose cardinality is the minimal period of  $x$ .*

*Proof.* Let  $p$  denote the minimal period of  $x$ . Then  $\mathcal{O}(x) = \{x, f(x), \dots, f^{p-1}(x)\}$  and  $|\mathcal{O}(x)| = p$ . Let  $y \in \alpha(x)$  (resp.  $y \in \omega(x)$ ). Then there exists a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that the sequence  $(f^{-n_k}(x))_{k \in \mathbb{N}}$  (resp.  $(f^{n_k}(x))_{k \in \mathbb{N}}$ ) converges to  $y$ . As  $\mathcal{O}(x)$  is finite and therefore discrete, we deduce that  $y \in \mathcal{O}(x)$ .

Conversely, let  $z \in \mathcal{O}(x)$ . Then there exists  $0 \leq i \leq p-1$  such that  $f^i(x) = z$ . Then the strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  defined by setting  $n_k := -i + p(k+1)$  (resp.  $n_k := i + pk$ ) for all  $k \in \mathbb{N}$  satisfies  $f^{-n_k}(x) = z$  (resp.  $f^{n_k}(x) = z$ ) for all  $k \in \mathbb{N}$ , showing that  $z \in \alpha(x)$  (resp.  $z \in \omega(x)$ ).  $\square$

Let  $(X, f)$  be a dynamical system. Let  $d$  be a compatible metric on  $X$ . Two points  $x, y \in X$  are said to be *unstably equivalent* (resp. *stably equivalent*, resp. *homoclinic*) if  $d(f^n(x), f^n(y)) \rightarrow 0$  as  $n \rightarrow -\infty$  (resp.  $n \rightarrow \infty$ , resp.  $|n| \rightarrow \infty$ ). Clearly, unstable equivalence (resp. stable equivalence, resp. homoclinicity) is an equivalence relation on  $X$ . By compactness of  $X$ , each of these equivalence relations does not depend on the choice of the compatible metric  $d$ . Given  $x \in X$ , we denote by  $W_f^u(x)$  (resp.  $W_f^s(x)$ , resp.  $W_f^h(x)$ ) the unstable (resp. stable, resp. homoclinic) equivalence class of  $x$ . Observe that  $W_f^h(x) = W_f^u(x) \cap W_f^s(x)$  and that  $f^n(W_f^u(x)) = W_f^u(f^n(x))$  (resp.  $f^n(W_f^s(x)) = W_f^s(f^n(x))$ , resp.  $f^n(W_f^h(x)) = W_f^h(f^n(x))$ ) for all  $x \in X$  and  $n \in \mathbb{Z}$ . Also,  $W_{f^{-1}}^u(x) = W_f^s(x)$  and  $W_{f^{-1}}^s(x) = W_f^u(x)$ . In particular,  $W_{f^{-1}}^h(x) = W_f^h(x)$ .

**Proposition 2.2.** *Let  $(X, f)$  be a dynamical system and let  $x, y \in X$ . Then the following hold:*

- (i) if  $x$  and  $y$  are unstably equivalent then  $\alpha(x) = \alpha(y)$ ;
- (ii) if  $x$  and  $y$  are stably equivalent then  $\omega(x) = \omega(y)$ ;
- (iii) if  $x$  and  $y$  are homoclinic then  $\alpha(x) = \alpha(y)$  and  $\omega(x) = \omega(y)$ .

*Proof.* Let  $d$  be a compatible metric on  $X$ . Suppose that  $x$  and  $y$  are unstably equivalent. Let  $z \in \alpha(x)$ . This means that there exists a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that the sequence  $(f^{-n_k}(x))_{k \in \mathbb{N}}$  converges to  $z$ . As  $x$  and  $y$  are unstably equivalent, the sequence  $(d(f^{-n}(x), f^{-n}(y)))_{n \in \mathbb{N}}$  converges to 0. Since

$$d(z, f^{-n_k}(y)) \leq d(z, f^{-n_k}(x)) + d(f^{-n_k}(x), f^{-n_k}(y))$$

by the triangle inequality, we deduce that  $d(z, f^{-n_k}(y))$  tends to 0 as  $k \rightarrow \infty$ . Therefore  $z \in \alpha(y)$ . This shows that  $\alpha(x) \subset \alpha(y)$ . By symmetry, we get  $\alpha(y) \subset \alpha(x)$ , so that  $\alpha(x) = \alpha(y)$ . This shows (i).

The proof of (ii) is similar.

Assertion (iii) follows from (i) and (ii) since homoclinicity implies both unstable and stable equivalence.  $\square$

The *category of dynamical systems* is the category whose objects consist of all dynamical systems  $(X, f)$  and whose morphisms are defined as follows. Given two dynamical systems  $(X, f)$  and  $(Y, g)$ , a *morphism*  $\varphi: (X, f) \rightarrow (Y, g)$  is a continuous map  $\varphi: X \rightarrow Y$  such that  $\varphi \circ f = g \circ \varphi$ . Thus, an endomorphism of  $(X, f)$  is a continuous map  $\tau: X \rightarrow X$  which commutes with  $f$ , i.e., such that  $\tau \circ f = f \circ \tau$ . We shall denote by  $\text{End}(X, f)$  the endomorphism monoid of a dynamical system  $(X, f)$ .

Observe that if  $\varphi: (X, f) \rightarrow (Y, g)$  is a morphism of dynamical systems, then  $(\varphi(X), g)$  is a subsystem of  $(Y, g)$ . A surjective morphism from  $(X, f)$  to  $(Y, g)$  is called a *factor map*. If there exists a factor map from  $(X, f)$  to  $(Y, g)$ , one says that  $(Y, g)$  is a *factor* of  $(X, f)$  and that  $(X, f)$  is an *extension* of  $(Y, g)$ . One says that  $(X, f)$  and  $(Y, g)$  are *topologically conjugate* if  $(X, f)$  and  $(Y, g)$  are isomorphic objects in the category of dynamical systems. Thus,  $(X, f)$  and  $(Y, g)$  are topologically conjugate if and only if there exists a homeomorphism  $\varphi: X \rightarrow Y$  such that  $\varphi \circ f = g \circ \varphi$ .

**Proposition 2.3.** *Let  $\varphi: (X, f) \rightarrow (Y, g)$  be a morphism of dynamical systems. Then the following hold:*

- (i) one has  $\varphi(\text{NW}(X)) \subset \text{NW}(Y)$ ;
- (ii) for every  $x \in X$ , one has  $\varphi(\alpha_f(x)) = \alpha_g(\varphi(x))$  and  $\varphi(\omega_f(x)) = \omega_g(\varphi(x))$ ;
- (iii) for every  $x \in X$ , one has  $\varphi(W_f^u(x)) \subset W_g^u(\varphi(x))$ ,  $\varphi(W_f^s(x)) \subset W_g^s(\varphi(x))$ , and  $\varphi(W_f^h(x)) \subset W_g^h(\varphi(x))$ .
- (iv) if  $(X, f)$  is non-wandering (resp. irreducible, resp. topologically transitive, resp. topologically mixing) then  $(\varphi(X), g)$  is non-wandering (resp. irreducible, resp. topologically transitive, resp. topologically mixing).

*Proof.* Assertion (i) follows, e.g., from Proposition 2.2.(iii) in [9].

To prove (ii), we first observe that  $\varphi \circ f^n = g^n \circ \varphi$  for all  $n \in \mathbb{Z}$  since  $\varphi \circ f = g \circ \varphi$ . Let now  $x \in X$ . Suppose first that  $y \in \varphi(\alpha_f(x))$ . This means that there exist  $z \in X$  with  $y = \varphi(z)$  and a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that

$f^{-n_k}(x)$  converges to  $z$  as  $k \rightarrow \infty$ . Then  $g^{-n_k}(\varphi(x)) = \varphi(f^{-n_k}(x))$  converges to  $\varphi(z) = y$  by continuity of  $\varphi$ . Therefore  $y \in \alpha_g(\varphi(x))$ . This shows that  $\varphi(\alpha_f(x)) \subset \alpha_g(\varphi(x))$ .

To prove the converse inclusion, suppose now  $y \in \alpha_g(\varphi(x))$ . This means that there exists a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that the sequence  $(g^{-n_k}(\varphi(x)))_{k \in \mathbb{N}}$  converges to  $y$ . By compactness of  $X$ , there exists a subsequence  $(m_i)_{i \in \mathbb{N}}$  of the sequence  $(n_k)_{k \in \mathbb{N}}$  such that the sequence  $(f^{-m_i}(x))_{i \in \mathbb{N}}$  converges to some point  $z \in X$ . We then have  $z \in \alpha_f(x)$ . On the other hand,

$$y = \lim_{i \rightarrow \infty} g^{-m_i}(\varphi(x)) = \lim_{i \rightarrow \infty} \varphi(f^{-m_i}(x)) = \varphi(z)$$

by continuity of  $\varphi$ . This shows that  $y \in \varphi(\alpha_f(x))$  and completes the proof that  $\varphi(\alpha_f(x)) = \alpha_g(\varphi(x))$ .

The proof of the second equality in (ii) is similar (it can also be deduced from the first equality applied to  $f^{-1}$  and the second equality in (2.1)).

Assertion (iii) follows, e.g., from Proposition 2.3 in [9].

If  $(X, f)$  is non-wandering, then  $(\varphi(X), g)$  is itself non-wandering by (i). On the other hand, if  $U$  and  $V$  are non-empty subsets of  $\varphi(X)$ , then  $U' := \varphi^{-1}(U)$  and  $V' := \varphi^{-1}(V)$  are non-empty open subsets of  $X$ . Moreover, if  $U' \cap f^n(V') \neq \emptyset$  for some  $n \geq 1$ , then

$$\emptyset \neq \varphi(U' \cap f^n(V')) = \varphi(\varphi^{-1}(U) \cap f^n(\varphi^{-1}(V))) = \varphi(\varphi^{-1}(U) \cap \varphi^{-1}(g^n(V))) \subset U \cap g^n(V),$$

so that  $U \cap g^n(V) \neq \emptyset$ . We deduce that if  $(X, f)$  is irreducible (resp. topologically mixing) then  $(\varphi(X), g)$  is irreducible (resp. topologically mixing). This completes the proof of (iv).  $\square$

In the trivial case when the phase space  $X$  of a dynamical system  $(X, f)$  is finite, the topology on  $X$  is discrete and we have the following straightforward characterization of irreducibility.

**Proposition 2.4.** *Let  $(X, f)$  be a dynamical system and suppose that  $X$  is finite with cardinality  $|X| = n$ . Then the following conditions are equivalent:*

- (a)  $(X, f)$  is irreducible;
- (b)  $(X, f)$  is topologically transitive;
- (c) every point of  $X$  is periodic with minimal period  $n$ ;
- (d) there exists a point of  $X$  which is periodic with minimal period  $n$ ;
- (e)  $(X, f)$  is topologically conjugate to the dynamical system  $(Y, g)$ , where  $Y := \mathbb{Z}/n\mathbb{Z}$  and  $g: Y \rightarrow Y$  is given by  $g(y) := y + 1$  for all  $y \in Y$ .

One says that a dynamical system  $(X, f)$  is *surjunctive* if every injective endomorphism of  $(X, f)$  is surjective and that it is *injunctive* if every surjective endomorphism of  $(X, f)$  is injective. One says that an endomorphism  $\tau$  of a dynamical system  $(X, f)$  is *pre-injective* if the restriction of  $\tau$  to every homoclinicity class of  $X$  is injective. One says that a dynamical system  $(X, f)$  has the *Moore property*, or that it is *Moore*, if every surjective endomorphism of  $(X, f)$  is pre-injective. One says that a dynamical system  $(X, f)$  has the *Myhill property*, or that it is *Myhill*, if every pre-injective endomorphism of  $(X, f)$  is surjective. Since

injectivity implies pre-injectivity, we always have the following implications:

$$(\text{Myhill} \implies \text{surjunctive}) \text{ and } (\text{injunctive} \implies \text{Moore}).$$

These implications are strict (see Example 8.3 and Example 8.6).

Surjunctivity, injunctivity, and the Moore property are clearly finiteness conditions in the sense that they are satisfied by all dynamical systems with finite phase space. By contrast, the Myhill property is not a finiteness condition (see Example 8.3).

**2.6. Shifts and subshifts.** Let  $A$  be a finite set, called the *alphabet*.

Consider the set  $A^{\mathbb{Z}}$  consisting of all bi-infinite sequences  $x = (x_i)_{i \in \mathbb{Z}}$ , where  $x_i \in A$  for all  $i \in \mathbb{Z}$ . An element of  $A^{\mathbb{Z}}$  is also called a *configuration* over the alphabet  $A$ . For  $x \in A^{\mathbb{Z}}$  and  $i, j \in \mathbb{Z}$  such that  $i \leq j$ , we define the word  $x_{[i,j]} \in A^{j-i+1}$  by  $x_{[i,j]} := x_i x_{i+1} \cdots x_j$ . We also define the left-infinite word  $x_{(-\infty, i]} \in A^{-\mathbb{N}}$  and the right-infinite word  $x_{[i, \infty)} \in A^{\mathbb{N}}$  by  $x_{(-\infty, i]} := \cdots x_{i-2} x_{i-1} x_i$  and  $x_{[i, \infty)} := x_i x_{i+1} x_{i+2} \cdots$ .

We equip  $A$  with its discrete topology and  $A^{\mathbb{Z}}$  with the topology of pointwise convergence. Thus, regarding  $A^{\mathbb{Z}}$  as the product of a family of copies of  $A$  indexed by  $\mathbb{Z}$ , the topology on  $A^{\mathbb{Z}}$  is the product topology. A base for the topology on  $A^{\mathbb{Z}}$  is formed by the *cylinders*

$$[w]_i := \{x \in A^{\mathbb{Z}} : x_{[i, i+|w|-1]} = w\},$$

where  $w$  runs over  $A^*$  and  $i$  over  $\mathbb{Z}$ . Note that every cylinder is *clopen* (i.e., both closed and open) in  $A^{\mathbb{Z}}$ .

One says that a word  $w \in A^*$  *appears* in  $x \in A^{\mathbb{Z}}$  if there exists  $i \in \mathbb{Z}$  such that  $x \in [w]_i$ .

The space  $A^{\mathbb{Z}}$  is compact, metrizable, and totally disconnected. A compatible metric  $d$  on  $A^{\mathbb{Z}}$  is obtained by setting, for all  $x, y \in A^{\mathbb{Z}}$ ,

$$(2.2) \quad d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ 2^{-\min\{i \in \mathbb{N} : (x_{-i}, x_i) \neq (y_{-i}, y_i)\}} & \text{if } x \neq y. \end{cases}$$

We also equip  $A^{\mathbb{Z}}$  with the homeomorphism  $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined by  $(\sigma(x))_i := x_{i+1}$  for all  $x \in A^{\mathbb{Z}}$  and  $i \in \mathbb{Z}$ . The homeomorphism  $\sigma$  is called the *shift map* and the dynamical system  $(A^{\mathbb{Z}}, \sigma)$  is called the *full shift* over the alphabet  $A$ .

Two sequences  $x, y \in A^{\mathbb{Z}}$  are unstably equivalent (resp. stably equivalent, resp. homoclinic) if and only if there exists an integer  $N \geq 0$  such that  $x_i = y_i$  for all  $i \in \mathbb{Z}$  with  $i \leq -N$  (resp.  $i \geq N$ , resp.  $|i| \geq N$ ).

A closed  $\sigma$ -invariant subset  $X \subset A^{\mathbb{Z}}$  is called a *subshift* over the alphabet  $A$ .

Let  $X \subset A^{\mathbb{Z}}$  be a subshift. The *language* of  $X$  is the subset  $L(X) \subset A^*$  consisting of all words  $w \in A^*$  such that  $w$  appears in some configuration  $x \in X$ .

A subshift  $X \subset A^{\mathbb{Z}}$  is non-wandering if and only if, for every  $u \in L(X)$ , there exists  $v \in A^*$  such that  $uvu \in L(X)$ . A non-empty subshift  $X \subset A^{\mathbb{Z}}$  is irreducible if and only if, for all  $u, v \in L(X)$ , there exists  $w \in A^*$  such that  $uwv \in L(X)$ , and it is topologically mixing if and only if, for all  $u, v \in L(X)$ , there exists an integer  $N \geq 1$  such that, for any integer  $n \geq N$ , there exists a word  $w \in A^n$  such that  $uwv \in L(X)$ .

A subset  $X \subset A^{\mathbb{Z}}$  is a subshift if and only if there exists a subset  $F \subset A^*$  such that  $X$  consists of all  $x \in A^{\mathbb{Z}}$  such that no element of  $F$  appears in  $x$ . Such a subset  $F$  is then

called a *defining set of forbidden words* for  $X$ . One says that a subshift  $X \subset A^{\mathbb{Z}}$  is of *finite type* if  $X$  admits a finite defining set of forbidden words.

Being of finite type is a topological conjugacy invariant for subshifts, i.e., if  $A, B$  are finite sets and  $X \subset A^{\mathbb{Z}}, Y \subset B^{\mathbb{Z}}$  are topologically conjugate subshifts, then  $X$  is of finite type if and only if  $Y$  is of finite type (see, e.g., [18, Theorem 2.1.10], [8, Exercise 1.54]).

The following result is a form of the *Curtis-Hedlund-Lyndon theorem*.

**Proposition 2.5.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift. Let  $\tau: X \rightarrow X$  be a map. Then the following conditions are equivalent:*

- (a)  $\tau$  is an endomorphism of  $X$ ;
- (b)  $\tau$  commutes with  $\sigma$  and there exists an integer  $m \geq 0$  and a map  $\mu: A^{2m+1} \rightarrow A$  such that  $\tau(x)_0 = \mu(x_{[-m,m]})$  for all  $x \in X$ .

*Proof.* Suppose (a). Then the map  $X \rightarrow A, x \mapsto \tau(x)_0$ , is continuous. It follows that there exists an integer  $m \geq 0$  such that if  $x, y \in X$  satisfy  $x_{[-m,m]} = y_{[-m,m]}$  then  $\tau(x)_0 = \tau(y)_0$ . We deduce that there exists a map  $\mu: A^{2m+1} \rightarrow A$  such that  $\tau(x)_0 = \mu(x_{[-m,m]})$  for all  $x \in X$ . This shows that (a) implies (b).

Conversely, suppose (b). Let  $x \in X$  and  $i \in \mathbb{Z}$ . Since  $\tau$  commutes with  $\sigma$ , we have  $\tau(x)_i = \tau(\sigma^i(x))_0$ . It follows that  $\tau(x)_i = \mu(x_{[-m+i,m+i]})$ . We deduce that, for every  $i \in \mathbb{Z}$ , the map  $X \rightarrow A, x \mapsto \tau(x)_i$  is continuous. This implies that the map  $\tau: X \rightarrow X$  itself is continuous. This shows that (b) implies (a).  $\square$

**2.7. Cantor-Bendixson decompositions and ranks.** A separable topological space  $X$  is said to be *Polish* if  $X$  admits a compatible metric  $d$  such that  $(X, d)$  is a complete metric space. Every compact metrizable space is Polish (see, e.g., [15, Proposition I.4.2]). A subset of a Polish space is Polish (for the relative topology) if and only if it is a  $G_\delta$ -subset, i.e., the intersection of a countable family of open subsets [15, Theorem I.3.11]. In particular, every open (resp. closed) subset of a Polish space is itself Polish.

Let  $X$  be a Polish space. By the *Cantor-Bendixson decomposition theorem* [15, Theorem I.6.4], there is a unique perfect closed subset  $P$  of  $X$  such that  $X \setminus P$  is countable. The subset  $P$  is called the *perfect kernel* of  $X$ . Observe that  $X$  is perfect (resp. countable) if and only if  $P = X$  (resp.  $P = \emptyset$ ).

Let  $X$  be a topological space. Recall that one says that a point  $x \in X$  is a *limit point* of a subset  $Y \subset X$  if every neighborhood of  $x$  contains a point of  $Y$  other than  $x$ . The set of all limit points of  $Y$  is called the *derived set* of  $Y$  and denoted by  $Y'$ . The set  $Y$  is closed in  $X$  if and only if  $Y' \subset Y$ . Note that  $X \setminus X'$  is the set of isolated points of  $X$ . Given an ordinal  $\alpha$ , the *derived set of order  $\alpha$*  of  $X$  is the closed subset  $X^{(\alpha)} \subset X$  defined, using transfinite induction, by setting  $X^{(0)} := X, X^{(\alpha+1)} := (X^{(\alpha)})'$  for every ordinal  $\alpha$ , and  $X^{(\lambda)} := \bigcap_{\alpha < \lambda} X^{(\alpha)}$  for every limit ordinal  $\lambda$ . There exists an ordinal  $\alpha_0$  such that  $X^{(\alpha)} = X^{(\alpha_0)}$  for every ordinal  $\alpha \geq \alpha_0$ . The smallest such  $\alpha_0$  is called the *Cantor-Bendixson rank* of  $X$  and denoted by  $\text{rk}_{\text{CB}}(X)$ .

In the case when  $X$  is Polish,  $\text{rk}_{\text{CB}}(X)$  is a countable ordinal and  $X^{(\text{rk}_{\text{CB}}(X))}$  is the perfect kernel of  $X$  [15, Theorem I.6.11]. A Polish space  $X$  is countable if and only if  $X^{(\text{rk}_{\text{CB}}(X))} = \emptyset$ .

If  $X$  is a non-empty countable compact metrizable space, then  $\text{rk}_{\text{CB}}(X)$  is a *successor ordinal*, i.e., there exists an ordinal  $\alpha$  such that  $\text{rk}_{\text{CB}}(X) = \alpha + 1$ . Moreover,  $\alpha$  is the smallest ordinal such that  $X^{(\alpha)}$  is finite. The *Mazurkiewicz-Sierpinski characteristic* of a countable compact metrizable space  $X$  is the pair  $(\alpha, n)$ , where  $\alpha$  is the smallest ordinal such that  $X^{(\alpha)}$  is finite and  $n := |X^{(\alpha)}|$ . The Mazurkiewicz-Sierpinski characteristic is a complete topological invariant for countable compact metrizable spaces, that is, two countable compact metrizable spaces are homeomorphic if and only if they have the same Mazurkiewicz-Sierpinski characteristic [19].

### 3. THE POSET OF IRREDUCIBLE COMPONENTS

In this section, we investigate the poset of irreducible components of a subshift of finite type. This is a finite poset which is a kind of skeleton for the subshift. We first describe this poset for vertex subshifts.

**3.1. Vertex subshifts.** Let  $G = (V, E)$  be a finite directed graph. The set  $X \subset V^{\mathbb{Z}}$ , consisting of all sequences  $x = (x_i)_{i \in \mathbb{Z}}$  such that  $x_i \in V$  and  $x_i x_{i+1} \in E$  for all  $i \in \mathbb{Z}$ , is a subshift of finite type over the alphabet  $V$  admitting  $F := V^2 \setminus E$  as a defining set of forbidden words. One says that  $X$  is the *vertex subshift* associated with  $G$ .

If  $\pi = v_0 v_1 \cdots v_{n-1} v_0$  is a cycle of length  $n$  in  $G$ , then the sequence  $x \in V^{\mathbb{Z}}$ , defined by  $x_i := v_j$  for all  $i \in \mathbb{Z}$  and  $j \in \{0, 1, \dots, n-1\}$  such that  $i \equiv j \pmod{n}$ , is a periodic configuration in  $X$  with period  $n$ . One says that  $x$  is the periodic configuration *associated with the cycle*  $\pi$ .

For  $W \subset V$ , the vertex subshift  $Y \subset W^{\mathbb{Z}}$  associated with the subgraph  $G_W$  of  $G$  induced by  $W$  is called the vertex subshift associated with  $W$ . Note that  $Y$  is also a subshift of finite type of  $V^{\mathbb{Z}}$  and that  $Y$  is irreducible whenever  $W$  is strongly connected.

If  $A$  is a finite set and  $X \subset A^{\mathbb{Z}}$  is a subshift of finite type, then there exists a finite directed graph  $G$  such that  $X$  is topologically conjugate to the vertex subshift associated with  $G$  (see, e.g., [18, Theorem 2.3.2]).

The following observation will be useful in the study of the topological properties of subshifts of finite type.

**Proposition 3.1.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the vertex subshift associated with  $G$ . Let  $\mathcal{K}$  denote the poset of path-components of  $G$  and let  $T \subset V$  denote the set of its transient vertices. Fix  $x \in X$ . Let  $\Gamma = \Gamma(x)$  denote the subset of  $\mathcal{K}$  consisting of all  $K \in \mathcal{K}$  such that there exists  $i \in \mathbb{Z}$  for which  $x_i \in K$ . Then  $\Gamma$  is a non-empty chain of  $\mathcal{K}$ . Moreover, writing  $\Gamma = \{K_0, K_1, K_2, \dots, K_{n-1}, K_n\}$  with  $K_0 \prec K_1 \prec K_2 \prec \cdots \prec K_{n-1} \prec K_n$  and  $n = |\Gamma| - 1 \geq 0$ , the following hold:*

- (i) *for each  $t \in T$ , there is at most one  $i \in \mathbb{Z}$  such that  $x_i = t$ ;*
- (ii) *the set of  $i \in \mathbb{Z}$  such that  $x_i \in T$  is finite;*
- (iii) *for each  $K \in \mathcal{K}$ , the set of  $i \in \mathbb{Z}$  such that  $x_i \in K$  is an interval of  $\mathbb{Z}$ ;*
- (iv) *if  $n = 0$ , then  $x_i \in K_0$  for all  $i \in \mathbb{Z}$ ;*
- (v) *if  $n \geq 1$ , then there exist unique integers*

$$b_0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{n-1} \leq b_{n-1} < a_n$$

in  $\mathbb{Z}$  such that one has, for every  $i \in \mathbb{Z}$ ,

$$(3.1) \quad x_i \in \begin{cases} K_0 & \text{if } i \in (-\infty, b_0], \\ K_j & \text{if } i \in [a_j, b_j] \text{ for some } j \in \{1, 2, \dots, n-1\}, \\ K_n & \text{if } i \in [a_n, \infty), \\ T & \text{otherwise.} \end{cases}$$

*Proof.* Let  $t \in T$  and suppose by contradiction that  $x_i = x_j = t$  for some  $i, j \in \mathbb{Z}$  with  $i < j$ . Then the word  $x_i x_{i+1} \cdots x_j$  is a cycle based at  $t$ , contradicting the fact that the vertex  $t$  is transient. This shows (i).

Assertion (ii) follows from (i) since the set  $T$  is finite.

If  $i \leq j$  are integers such that  $x_i, x_j \in K$  for some  $K \in \mathcal{K}$ , then  $x_k \in K$  for all integers  $k \in [i, j]$ . Indeed, if  $i < k < j$ , then  $x_i x_{i+1} \cdots x_k$  is a path from  $x_i$  to  $x_k$  and  $x_k x_{k+1} \cdots x_j$  is a path from  $x_k$  to  $x_j$ . This shows (iii).

We know that  $\mathcal{K}$  is a partition of  $R = V \setminus T$ . Thus, given  $i \in \mathbb{Z}$ , either  $x_i \in T$  or  $x_i \in K$  for some unique  $K \in \mathcal{K}$ . On the other hand, if  $K, K' \in \mathcal{K}$  are such that  $x_i \in K$  and  $x_j \in K'$  for some integers  $i < j$ , then  $K \preceq K'$ , since  $x_i x_{i+1} \cdots x_j$  is a path in  $G$ . As the poset  $\mathcal{K}$  is finite, combining these observations with (ii) and (iii), we deduce that  $\Gamma$  is a non-empty chain and we get (iv) and (v).  $\square$

Let  $X \subset V^{\mathbb{Z}}$  denote the vertex subshift associated with a finite directed graph  $G = (V, E)$  and let  $x \in X$ . With the notation introduced in Proposition 3.1, one says that  $\Gamma$  is the *chain of path-components* of  $x$  and that  $K_0$  (resp.  $K_n$ ) is the *initial* (resp. *terminal*) path-component of  $x$ .

**Proposition 3.2.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the vertex subshift associated with  $G$ . Let  $x \in X$  and denote by  $K$  (resp.  $K'$ ) the initial (resp. terminal) path-component of  $x$ . Let  $C$  (resp.  $C'$ ) denote the irreducible subshift of  $V^{\mathbb{Z}}$  associated with  $K$  (resp.  $K'$ ). Then one has*

$$(3.2) \quad \alpha(x) \subset C \quad \text{and} \quad \omega(x) \subset C'.$$

Moreover, the following conditions are equivalent:

- (a)  $C = C'$ ;
- (b)  $K = K'$ ;
- (c)  $x \in C$ ;
- (d)  $x \in C'$ ;
- (e)  $x \in \text{NW}(X)$ .

*Proof.* Let  $y \in \alpha(x)$  and let  $U$  be a neighborhood of  $y$  in  $X$ . Choose  $k \in \mathbb{N}$  so that

$$U' := \{z \in X : z_{[-k, k]} = y_{[-k, k]}\} \subset U.$$

By Assertions (iv) and (v) in Proposition 3.1, there exists  $b_0 = b_0(x) \in \mathbb{Z}$  such that  $x_i \in K$  for all  $i \leq b_0$ . Since  $y \in \alpha(x)$ , there exists  $m \in \mathbb{N}$  such that  $k - m \leq b_0$  and  $\sigma^{-m}(x) \in U'$ . This implies  $y_i = x_{i-m} \in K$  for all  $-k \leq i \leq k$ . Since  $y_{-k} y_{-k+1} \cdots y_k$  is a path in  $K$  and  $K$  is strongly connected, we can find  $z \in C$  such that  $z_{[-k, k]} = y_{[-k, k]}$ . We then have

$z \in U' \subset U$ . Thus, every neighborhood of  $y$  in  $X$  meets  $C$ . As  $C$  is closed in  $X$ , we deduce that  $y \in C$ . This shows  $\alpha(x) \subset C$ .

The proof of  $\omega(x) \subset C'$  is similar.

This shows (3.2).

Let us prove now the equivalence of Conditions (a)—(e).

The elements of  $\mathcal{K}$  form a partition of the set of recurrent vertices. We deduce that (a) and (b) are equivalent.

If  $x \in C$ , then  $x_i \in K$  for all  $i \in \mathbb{Z}$ . This implies that  $K$  is both the initial and the terminal path-component of  $x$ . Therefore  $K = K'$ . Conversely, suppose now  $K = K'$ . Then  $x_i \in K$  for all  $i \in \mathbb{Z}$  by Proposition 3.1.(iv). Hence  $x \in C$ . This shows that (c) and (a) are equivalent.

The proof of the equivalence of (d) and (a) is similar.

Since the subgraph  $K$  is strongly connected, the subshift  $C$  is irreducible and hence non-wandering. It follows that if  $x \in C$  then  $x \in \text{NW}(X)$ . Therefore (c) implies (e).

Suppose now  $K \neq K'$ . With the notation of Proposition 3.1.(iv), we have  $K_0 = K \prec K' = K_n$ . Consider the open neighborhood  $U$  of  $x$  in  $X$  defined by

$$U := \{y \in X : y_{[b_0, a_n]} = x_{[b_0, a_n]}\}.$$

We deduce from Proposition 3.1.(iv) that the sets  $\sigma^m(U)$ ,  $m \geq 0$ , are all disjoint. Therefore  $x \notin \text{NW}(X)$ . This shows that (e) implies (b) and completes the proof of the equivalence of (a)—(e).  $\square$

**Proposition 3.3.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the subshift of finite type associated with  $G$ . Let  $\mathcal{K}$  denote the poset of path-components of  $G$ . For each  $K \in \mathcal{K}$ , denote by  $C_K \subset V^{\mathbb{Z}}$  the subshift of finite type associated with  $K$ . Then the following hold:*

- (i) *each  $C_K$ ,  $K \in \mathcal{K}$ , is an irreducible subshift of finite type;*
- (ii) *the subshifts  $C_K$ ,  $K \in \mathcal{K}$ , are pairwise disjoint;*
- (iii)  $\text{NW}(X) = \bigcup_{K \in \mathcal{K}} C_K$ ;
- (iv)  $\text{NW}(X)$  *is a subshift of finite type;*
- (v) *given  $K, K' \in \mathcal{K}$ , one has  $K \preceq K'$  if and only if there exists  $x \in X$  such that  $\alpha(x) \subset C_K$  and  $\omega(x) \subset C_{K'}$ .*

*Proof.* Assertion (i) follows from the fact that each  $K \in \mathcal{K}$  is strongly connected (this has been already observed in the proof of Proposition 3.2).

For every  $x \in C_K$ ,  $K \in \mathcal{K}$ , we have  $\{x_i : i \in \mathbb{Z}\} \subset K$ . As the elements of  $\mathcal{K}$  are pairwise disjoint, Assertion (ii) follows.

Assertion (iii) follows from the equivalence between Conditions (c) and (e) in Proposition 3.2.

By (iii),  $\text{NW}(X)$  is the vertex subshift associated with the subgraph of  $G$  whose set of vertices is  $\bigcup_{K \in \mathcal{K}} K$ . This shows (iv).

Let  $K, K' \in \mathcal{K}$ . Choose vertices  $v \in K$  and  $v' \in K'$ . Suppose first  $K \preceq K'$ . This means that there exists a path  $v_0 \cdots v_n$  in  $G$  of length  $n \geq 1$  from  $v$  to  $v'$ . Since the vertices  $v$  and  $v'$  are recurrent, we can find a cycle  $c_0 \cdots c_k$  based at  $v$  and a cycle  $c'_0 \cdots c'_k$  based at

$v'$ . Consider the sequence  $x \in V^{\mathbb{Z}}$  defined by

$$x_i := \begin{cases} c_j & \text{if } i \leq 0 \text{ and } i \equiv j \pmod{k}, \\ v_i & \text{if } 0 \leq i \leq n \\ c'_j & \text{if } n \leq i \text{ and } i - n \equiv j \pmod{k'}. \end{cases}$$

Then  $K$  (resp.  $K'$ ) is the initial (resp. terminal) path-component of  $x$ . Therefore  $\alpha(x) \subset C_K$  and  $\omega(x) \subset C_{K'}$  by (3.2).

Conversely, suppose that there exists  $x \in X$  such that  $\alpha(x) \subset C_K$  and  $\omega(x) \subset C_{K'}$ . Then  $K$  (resp.  $K'$ ) is the initial (resp. terminal) path-component of  $x$  by (3.2). It follows that there exist integers  $m, n$  with  $m < n$  such that  $x_i \in K$  for all  $i \leq m$  and  $x_i \in K'$  for all  $n \leq i$ . Then  $x_m \cdots x_n$  is a path in  $G$  from  $x_m \in K$  to  $x_n \in K'$ . Therefore  $K \preceq K'$ . This completes the proof of (v).  $\square$

### 3.2. Irreducible components of subshifts of finite type.

**Proposition 3.4.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then the following hold:*

- (i) *the subshift  $\text{NW}(X)$  is of finite type;*
- (ii) *if  $\mathcal{C}$  is a finite partition of  $\text{NW}(X)$  into irreducible subshifts and  $Z \subset X$  is an irreducible subshift, then there exists a unique  $C \in \mathcal{C}$  such that  $Z \subset C$ ;*
- (iii) *there exists a unique finite partition  $\mathcal{C}$  of  $\text{NW}(X)$  into irreducible subshifts;*
- (iv) *each subshift  $C \in \mathcal{C}$  is of finite type;*
- (v) *given  $x \in X$ , there exist a unique  $C_\alpha(x) \in \mathcal{C}$  and a unique  $C_\omega(x) \in \mathcal{C}$  such that  $\alpha(x) \subset C_\alpha(x)$  and  $\omega(x) \subset C_\omega(x)$ ;*
- (vi) *given  $x \in X$  and  $C \in \mathcal{C}$ , one has  $C_\alpha(x) = C_\omega(x) = C$  if and only if  $x \in C$ ;*
- (vii) *given  $x \in X$ , one has  $x \in \text{NW}(X)$  if and only if  $C_\alpha(x) = C_\omega(x)$ ;*
- (viii) *the relation  $\preceq$  on  $\mathcal{C}$ , defined by writing  $C \preceq C'$  for  $C, C' \in \mathcal{C}$  if there exists  $x \in X$  such that  $C_\alpha(x) = C$  and  $C_\omega(x) = C'$ , is a partial ordering on  $\mathcal{C}$ .*

*Proof.* Since every subshift of finite type is topologically conjugate to a vertex subshift, we may assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ .

Assertion (i) is then Assertion (iv) in Proposition 3.3.

To prove (ii), suppose that  $\mathcal{C}$  is a finite partition of  $\text{NW}(X)$  into irreducible subshifts and let  $Z \subset X$  be an irreducible subshift. Then  $Z \subset \text{NW}(X)$  since every irreducible dynamical system is non-wandering. As  $\mathcal{C}$  is a finite partition of  $\text{NW}(X)$  into closed subsets, each element of  $\mathcal{C}$  is open in  $\text{NW}(X)$ . We deduce that the sets  $Z \cap C_K$ ,  $K \in \mathcal{K}$ , are open  $\sigma$ -invariant subsets of  $Z$ . Since  $Z$  is irreducible, it follows that there exists a unique  $C \in \mathcal{C}$  such that  $Z \subset C$ . This shows (ii).

To prove (iii), consider the set  $\mathcal{C} := \{C_K : K \in \mathcal{K}\}$ , where  $\mathcal{K}$  denotes the set of path-components of  $G$  and  $C_K$  denotes the vertex subshift associated with  $K \in \mathcal{K}$ . We know that  $\mathcal{C}$  is a finite partition of  $\text{NW}(X)$  into irreducible subshifts (cf. Assertions (i), (ii), and (iii) in Proposition 3.3). This shows existence in (iii).

To prove uniqueness, suppose that  $\mathcal{C}'$  is another finite partition of  $\text{NW}(X)$  into irreducible subshifts. By (ii), there exist a map  $f: \mathcal{C} \rightarrow \mathcal{C}'$  such that  $C \subset f(C)$  for all  $C \in \mathcal{C}$  and

a map  $g: \mathcal{C}' \rightarrow \mathcal{C}$  such that  $C' \subset g(C')$  for all  $C' \in \mathcal{C}'$ . We then have  $C \subset g(f(C))$  and hence  $C = g(f(C))$  for all  $C \in \mathcal{C}$ . This implies that  $f$  is bijective and  $\mathcal{C} = \mathcal{C}'$  follows. This completes the proof of (iii).

Assertion (iv) follows from Proposition 3.3.(i).

Assertions (v), (vi), and (vii) follow from Proposition 3.2.

Assertion (viii) follows from Proposition 3.3.(vi).  $\square$

With the notation of Proposition 3.4, the subshifts  $C, C \in \mathcal{C}$ , are called the *irreducible components* of  $X$ . As  $\mathcal{C}$  is a poset, it makes sense to speak of an isolated (resp. intermediate, resp. extremal, resp. maximal, resp. minimal) irreducible component of  $X$ .

From the results of Proposition 3.4, we get the following characterization of non-wandering subshifts of finite type.

**Corollary 3.5.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then  $X$  is non-wandering if and only if  $X$  is the union of its isolated irreducible components.*

**Remark 3.6.** Suppose that  $X \subset V^{\mathbb{Z}}$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . Let  $\mathcal{K}$  denote the set of path-components of  $G$ . For each  $K \in \mathcal{K}$ , denote by  $C_K \subset V^{\mathbb{Z}}$  the subshift of finite type associated with  $K$ . It follows from Proposition 3.3 that the set of irreducible components of  $X$  is  $\mathcal{C} = \{C_K : K \in \mathcal{K}\}$  and that the map  $\mathcal{K} \rightarrow \mathcal{C}$ ,  $K \mapsto C_K$ , is a poset isomorphism. One says that  $K$  (resp.  $C_K$ ) is the path-component (resp. irreducible component) *associated with*  $C_K$  (resp.  $K$ ). Note also that, for every  $x \in X$ , if the initial (resp. terminal) path-component of  $x$  is  $K$  (resp.  $K'$ ), then one has  $C_\alpha(x) = C_K$  (resp.  $C_\omega(x) = C_{K'}$ ).

**Proposition 3.7.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the associated vertex subshift. Let  $K$  be a path-component of  $G$  and let  $C$  denote the irreducible component of  $X$  associated with  $K$ . Then the following conditions are equivalent:*

- (a)  $C$  is finite with cardinality  $|C| = n$ ;
- (b) there exists a simple cycle  $v_0v_1 \cdots v_n$  in  $G$  such that  $K = \{v_0, v_1, \dots, v_{n-1}\}$  and  $E \cap K^2 = \{v_rv_{r+1} : 0 \leq r \leq n-1\}$ .

*Proof.* Suppose (a). Let  $v_0v_1 \cdots v_m$  be a cycle in  $K$  of minimal length  $m \geq 1$ . Note that such a cycle is simple by minimality. Consider the associated periodic configuration  $x \in C$ , i.e., the configuration defined by  $x_i := v_j$  if  $i \in \mathbb{Z}$  and  $i \equiv j \pmod{m}$ . Then  $x$  is periodic with minimal period  $m$ . By applying Proposition 2.4, we deduce that  $\mathcal{O}(x) = C$  and  $m = n$ . Let  $v \in K$ . As  $v$  is recurrent, there exists a cycle  $v'_0v'_1 \cdots v'_{m'}$  based at  $v$ . Consider the associated periodic configuration  $x'$  defined by setting  $x'_i := v'_j$  if  $i \in \mathbb{Z}$  and  $i \equiv j \pmod{m'}$ . As  $x' \in C = \mathcal{O}(x)$ , there exists  $k \in \{0, 1, \dots, n-1\}$  such that  $x' = \sigma^k(x)$ . We deduce that  $v = v'_0 = x'_0 = (\sigma^k(x))_0 = x_k = v_k$ . This shows that  $K = \{v_0, v_1, \dots, v_{n-1}\}$ .

We have  $v_rv_{r+1} \in E$  for all  $0 \leq r \leq n-1$  since  $v_0v_1 \cdots v_n$  is a cycle in  $G$ . Conversely, suppose that  $v, w \in K$  are such that  $vw \in E$ . As  $K$  is a path-component, there exists a path  $w_1 \cdots w_p$  in  $K$  going from  $w$  to  $v$ . Then, setting  $w_0 := v$ , the word  $w_0w_1 \cdots w_p$  is a cycle based at  $v$  in  $K$ . Consider the associated periodic configuration  $z$  defined by  $z_i := w_j$  if  $i \in \mathbb{Z}$  and  $i \equiv j \pmod{p}$ . We have  $z \in C = \mathcal{O}(x)$ . Consequently, there exists

$r \in \{0, 1, \dots, n-1\}$  such that  $z = \sigma^r(x)$ . Then  $v = w_0 = z_0 = x_r$  and  $w = w_1 = z_1 = x_{r+1}$ . This shows that  $E \cap K^2 = \{v_r v_{r+1} : 0 \leq r \leq n-1\}$ . The implication (a)  $\implies$  (b) follows.

Conversely, suppose (b). Consider the configuration  $x \in C$  defined by  $x_i := v_j$  if  $i \in \mathbb{Z}$  and  $i \equiv j \pmod n$ . Note that  $x$  is periodic with minimal period  $n$  since  $v_0 v_1 \cdots v_{n-1}$  is a simple cycle. Let now  $y \in C$ . By (b), for every  $i \in \mathbb{Z}$ , there exists  $r \in \{0, 1, \dots, n-1\}$  such that  $y_i y_{i+1} = v_r v_{r+1}$ . Denoting by  $k$  the unique element in  $\{0, 1, \dots, n-1\}$  such that  $y_0 = v_k$ , we deduce, by induction on  $|i|$ , that  $y_i = v_j$  for every  $i \in \mathbb{Z}$ , where  $j \in \{0, 1, \dots, n-1\}$  satisfies  $k + i \equiv j \pmod n$ . Thus,  $y = \sigma^k(x) \in \mathcal{O}(x)$ . It follows that  $C = \mathcal{O}(x)$ , so that  $|C| = |\mathcal{O}(x)| = n$ . This shows that (b) implies (a).  $\square$

**Corollary 3.8.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the associated vertex subshift. Let  $K$  be a path-component of  $G$  and let  $C$  denote the irreducible component of  $X$  associated with  $K$ . Let  $x \in X$  and consider the interval  $I \subset \mathbb{Z}$  consisting of all  $i \in \mathbb{Z}$  such that  $x_i \in K$  (cf. Proposition 3.1.(iii)). Suppose that  $C$  is finite and let  $i_0 \in I$ . Then, the restriction of  $x$  to  $I$  is entirely determined by  $x_{i_0}$ .*

*Proof.* By virtue of Proposition 3.7, there exists a simple cycle  $v_0 v_1 \cdots v_n$  in  $G$  such that  $K = \{v_0, v_1, \dots, v_{n-1}\}$  and  $E \cap K^2 = \{v_r v_{r+1} : 0 \leq r \leq n-1\}$ . We deduce that there exists a unique element  $r \in \{0, 1, \dots, n-1\}$  such that  $x_{i_0} = v_r$ . Moreover, for every  $i \in I$ , we then have  $x_i = v_j$ , where  $j \in \{0, 1, \dots, n-1\}$  is given by  $j \equiv r + i - i_0 \pmod n$ .  $\square$

**Proposition 3.9.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then every infinite irreducible component of  $X$  contains infinitely many periodic configurations.*

*Proof.* Let  $C$  be an irreducible component of  $X$ . Since  $C$  is an irreducible subshift of finite type, its periodic configurations are dense in  $C$  (see, e.g., [18, Exercise 6.1.12] or [8, Exercise 3.35.(a)]). We deduce that  $C$  is finite if it contains only finitely many periodic configurations.  $\square$

Every irreducible component  $C$  of a subshift of finite type  $X$  is closed in  $X$ . By the following result,  $C$  is open (and hence clopen) in  $X$  if and only if  $C$  is isolated in the poset of irreducible components of  $X$ .

**Proposition 3.10.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Let  $\mathcal{C}$  denote the poset of irreducible components of  $X$  and let  $C \in \mathcal{C}$ . Then the following conditions are equivalent:*

- (a)  $C$  is isolated in the poset  $\mathcal{C}$ ;
- (b)  $C$  is an open subset of  $X$ .

*Proof.* Suppose that  $C$  is not isolated in  $\mathcal{C}$ . This means that there exists  $C' \in \mathcal{C}$  such that either  $C \prec C'$  or  $C' \prec C$ . Suppose that  $C \prec C'$  (the case when  $C' \prec C$  is analogous). It follows that there exists a configuration  $x \in X$  such that  $C_\alpha(x) = C$  and  $C_\omega(x) = C'$ . Then  $x \notin C$  and there exists a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  of non-negative integers such that the sequence  $(\sigma^{-n_k}(x))_{k \in \mathbb{N}}$  converges to some point in  $\alpha(x) \subset C_\alpha(x) = C$ . As  $\sigma^n(x) \notin C$  for all  $n \in \mathbb{Z}$  since  $C$  is  $\sigma$ -invariant, this implies that  $C$  is not open in  $X$ . This shows that (b) implies (a).

Conversely, suppose that  $C$  is isolated in  $\mathcal{C}$  and let us show that  $C$  is open in  $X$ . We may assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . Denote by  $K$  the path-component of  $G$  associated with  $C$ . Let  $x \in C$ . Then  $x_i \in K$  for all  $i \in \mathbb{Z}$ . In particular,  $x_0 \in K$ . Consider the open neighborhood  $U$  of  $x$  in  $X$  consisting of all configurations in  $X$  which take the same value as  $x$  at 0. If  $y \in U$ , then  $y_0 \in K$ . As  $K$  is isolated in the poset of path-components of  $G$ , we then deduce from Proposition 3.1 that  $y_i \in K$  for all  $i \in \mathbb{Z}$ . It follows that  $y \in C$ . Therefore  $U \subset C$ . This shows that  $C$  is open in  $X$ .  $\square$

### 3.3. Characterization of isolated configurations.

**Proposition 3.11.** *Let  $A$  be a finite set, let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type, and let  $x \in X$ . Denote by  $\mathcal{C}$  the poset of irreducible components of  $X$ . Then the following conditions are equivalent:*

- (a) *the configuration  $x$  is isolated in  $X$ ;*
- (b) *the irreducible components  $C_\alpha(x)$  and  $C_\omega(x)$  are both finite, with  $C_\alpha(x)$  minimal and  $C_\omega(x)$  maximal in  $\mathcal{C}$ .*

*Proof.* We may assume that  $X$  is the vertex shift associated with a finite directed graph  $G = (V, E)$ . To simplify, let us set  $C := C_\alpha(x)$  and  $C' := C_\omega(x)$ . Let  $K$  (resp.  $K'$ ) denote the path-component of  $G$  associated with  $C$  (resp.  $C'$ ). Since  $K$  (resp.  $K'$ ) is the initial (resp. terminal) path-component of  $x$ , there exist  $a, b \in \mathbb{Z}$  with  $a \leq b$  such that  $x_i \in K$  (resp.  $x_i \in K'$ ) for all  $i \leq a$  (resp.  $i \geq b$ ).

Let us assume that Condition (b) fails to hold. Let us show that  $x$  is not isolated in  $X$ . Suppose first that  $C$  is infinite (the case where  $C'$  is infinite is similar). Let  $M$  be a positive integer and set  $i_0 := \min(a, -M)$ . Given  $y \in \text{Per}(C)$ , we construct a configuration  $z = z(y) \in V^{\mathbb{Z}}$  as follows. Since  $y_0, x_{i_0} \in K$  and  $K$  is a path-component of  $G$ , there exist an integer  $k \geq 1$  and a path  $v_0 v_1 \cdots v_k$  of length  $k$  in  $K$  going from  $v_0 = y_0$  to  $v_k = x_{i_0}$ . Denoting by  $p(y)$  the minimal period of  $y$ , we then set, for every  $i \in \mathbb{Z}$ ,

$$(3.3) \quad z_i := \begin{cases} y_j & \text{if } i \leq i_0 - k \text{ and } i \equiv i_0 - k + j \pmod{p(y)} \\ v_{i-i_0+k} & \text{if } i_0 - k \leq i \leq i_0 \\ x_i & \text{if } i \geq i_0. \end{cases}$$

We clearly have  $z \in X$  and  $z_{[-M, M]} = x_{[-M, M]}$ . Moreover, the  $\alpha$ -limit set  $\alpha(z)$  of  $z$  equals  $\alpha(y) = \mathcal{O}(y)$  and, in particular,  $|\alpha(z)| = |\alpha(y)| = |\mathcal{O}(y)| = p(y)$  (cf. Proposition 2.1).

Since  $C$  is infinite, it contains infinitely many periodic configurations by Proposition 3.9. Thus, there exist  $y, y' \in \text{Per}(C)$  whose minimal periods  $p(y)$  and  $p(y')$  are distinct. Denote by  $z = z(y)$  and  $z' = z(y')$  the configurations in  $X$  associated with  $y$  and  $y'$  via (3.3). As  $|\alpha(z)| = p(y) \neq p(y') = |\alpha(z')|$ , we have  $z \neq z'$ . Therefore, we have  $x \neq z$  or  $x \neq z'$ . Since the positive integer  $M$  was arbitrary and the three configurations  $x, z$  and  $z'$  satisfy  $x_{[-M, M]} = z_{[-M, M]} = z'_{[-M, M]}$ , we deduce that  $x$  is not isolated in  $X$ .

Suppose now that the irreducible component  $C$  is not minimal (the case where  $C'$  is not maximal is similar). Let  $M$  be a positive integer and, as before, set  $i_0 := \min(a, -M)$ . As  $C$  is not minimal, there exists an irreducible component  $C''$  of  $X$  such that  $C'' \prec C$ . Let

$y \in C''$ . Denoting by  $K''$  the path-component of  $G$  associated with  $C''$ , we have  $K'' \prec K$ . Consequently, there exists an integer  $k \geq 1$  and a path  $v_0 v_1 \cdots v_k$  of length  $k$  in  $G$  going from  $v_0 = y_0$  to  $v_k = x_{i_0}$ . Consider the configuration  $z \in V^{\mathbb{Z}}$  defined, for all  $i \in \mathbb{Z}$ , by

$$z_i := \begin{cases} y_{i-i_0+k} & \text{if } i \leq i_0 - k \\ v_{i-i_0+k} & \text{if } i_0 - k \leq i \leq i_0 \\ x_i & \text{if } i \geq i_0. \end{cases}$$

Note that  $z \in X$ . We have  $z \neq x$  since  $C_\alpha(z) = C'' \neq C = C_\alpha(x)$ . As  $z_{[-M,M]} = x_{[-M,M]}$  and the positive integer  $M$  was arbitrary, this shows that  $x$  is not isolated. The implication (a)  $\implies$  (b) follows.

Conversely, assume (b). Let us show that  $x$  is isolated in  $X$ . Let  $y \in X$  such that  $y_{[a,b]} = x_{[a,b]}$ . Using the fact that  $K$  is minimal (resp.  $K'$  is maximal), we get  $x_i, y_i \in K$  (resp.  $x_i, y_i \in K'$ ) for all  $i \leq a$  (resp.  $i \geq b$ ). As  $C$  (resp.  $C'$ ) is finite and  $x_a = y_a$  (resp.  $x_b = y_b$ ) we deduce from Corollary 3.8 that  $x_i = y_i$  for all  $i \leq a$  (resp.  $i \geq b$ ). Since  $x_{[a,b]} = y_{[a,b]}$ , we have that  $x = y$ . As the set of configurations in  $X$  which coincide with  $x$  on  $[a, b]$  is a neighborhood of  $x$ , we conclude that  $x$  is isolated in  $X$ . This proves (b)  $\implies$  (a).  $\square$

**Corollary 3.12.** *Let  $A$  be a finite set, let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type, and let  $C$  be an infinite irreducible component of  $X$ . Then  $C$  is a perfect space. Consequently,  $C$  is homeomorphic to the Cantor set and hence uncountable.*

**Proposition 3.13.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Then  $X$  is countable if and only if every irreducible component of  $X$  is finite.*

*Proof.* If  $X$  contains an infinite component then  $X$  is uncountable by Corollary 3.12. Thus, the condition is necessary.

Conversely, suppose that every irreducible component of  $X$  is finite and let us show that  $X$  is countable. We may assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . Let  $x \in X$ . Set  $C := C_\alpha(x)$  and  $C' := C_\omega(x)$ . If  $C = C'$  then  $x \in C$ . Otherwise, we have  $C \prec C'$  and if the integers  $b_0 < a_n$  are as in Proposition 3.1, it follows from Corollary 3.8 that  $x_{(-\infty, b_0]}$  (resp.  $x_{[a_n, \infty)}$ ) is entirely determined by the value of  $x_{b_0}$  (resp.  $x_{a_n}$ ). We deduce that there are only finitely many  $x \in X$  for a fixed pair  $(b_0, a_n)$ . This implies that  $X$  is countable.  $\square$

### 3.4. Action of endomorphisms on irreducible components.

**Proposition 3.14.** *Let  $A$  be a finite set, let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type, and let  $\mathcal{C}$  denote the poset of irreducible components of  $X$ . Suppose that  $\tau: X \rightarrow X$  is an endomorphism of  $X$ . Then the following hold:*

- (i) *there exists a unique map  $\rho = \rho_\tau: \mathcal{C} \rightarrow \mathcal{C}$  such that, for every  $C \in \mathcal{C}$ , one has  $\tau(C) \subset \rho(C)$ ;*
- (ii) *for every  $x \in X$ , one has  $\rho(C_\alpha(x)) = C_\alpha(\tau(x))$  and  $\rho(C_\omega(x)) = C_\omega(\tau(x))$ ;*
- (iii) *the map  $\rho: \mathcal{C} \rightarrow \mathcal{C}$  is a poset endomorphism of  $\mathcal{C}$ .*

Moreover, the map  $\text{End}(X, \sigma) \rightarrow \text{End}(\mathcal{C}, \preceq)$ ,  $\tau \mapsto \rho_\tau$ , is a monoid morphism.

*Proof.* Let  $C \in \mathcal{C}$ . As  $\tau$  is an endomorphism of  $(X, \sigma)$ , we deduce from Proposition 2.3.(iv) that  $\tau(C) \subset X$  is an irreducible subshift. It then follows from Proposition 3.4.(iii) that there exists a unique  $C' \in \mathcal{C}$  such that  $\tau(C) \subset C'$ . This proves (i).

Let  $x \in X$ . We have  $\tau(\alpha(x)) = \alpha(\tau(x))$  and  $\tau(\omega(x)) = \omega(\tau(x))$  by Proposition 2.3.(ii). We deduce that

$$\tau(\alpha(x)) = \alpha(\tau(x)) \subset C_\alpha(\tau(x))$$

and

$$\tau(\omega(x)) = \omega(\tau(x)) \subset C_\omega(\tau(x)).$$

On the other hand, we have  $\emptyset \neq \alpha(\tau(x)) \subset C_\alpha(\tau(x))$  and  $\emptyset \neq \omega(\tau(x)) \subset C_\omega(\tau(x))$ . Using uniqueness in (i), we deduce that  $\tau(C_\alpha(x)) \subset C_\alpha(\tau(x))$  and  $\tau(C_\omega(x)) \subset C_\omega(\tau(x))$ . This shows (ii).

To prove (iii), suppose that  $C, C' \in \mathcal{C}$  are such that  $C \preceq C'$ . This means that there exists  $x \in X$  such that  $\alpha(x) \subset C$  and  $\omega(x) \subset C'$ . Using (ii), we get

$$\alpha(\tau(x)) = \tau(\alpha(x)) \subset \tau(C) \subset \rho(C)$$

and

$$\omega(\tau(x)) = \tau(\omega(x)) \subset \tau(C') \subset \rho(C').$$

We deduce that  $\rho(C) \preceq \rho(C')$ . This shows (iii).

We clearly have  $\rho_{\text{Id}_X} = \text{Id}_{\mathcal{C}}$ . On the other hand, let  $\tau_1, \tau_2 \in \text{End}(X, \sigma)$ . For every  $C \in \mathcal{C}$ , we have

$$(\tau_1 \tau_2)(C) = \tau_1(\tau_2(C)) \subset \rho_{\tau_1}(\tau_2(C)) \subset \rho_{\tau_1}(\rho_{\tau_2}(C)) = (\rho_{\tau_1} \rho_{\tau_2})(C).$$

Therefore  $\rho_{\tau_1 \tau_2} = \rho_{\tau_1} \rho_{\tau_2}$ . This shows that  $\tau \mapsto \rho_\tau$  is a monoid morphism from  $\text{End}(X, \sigma)$  to  $\text{End}(\mathcal{C}, \preceq)$ .  $\square$

**Proposition 3.15.** *Let  $A$  be a finite set, let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type, and let  $\mathcal{C}$  denote the poset of irreducible components of  $X$ . Suppose that  $\tau: X \rightarrow X$  is a surjective endomorphism of  $X$  and let  $\rho = \rho_\tau: \mathcal{C} \rightarrow \mathcal{C}$  denote the poset endomorphism of  $\mathcal{C}$  associated with  $\tau$ . Then the following hold:*

- (i) *the map  $\rho: \mathcal{C} \rightarrow \mathcal{C}$  is a poset automorphism of  $\mathcal{C}$ ;*
- (ii) *one has  $\tau^{-1}(C) = \rho^{-1}(C)$  for each  $C \in \mathcal{C}$ ;*
- (iii) *one has  $\tau(C) = \rho(C)$  for each  $C \in \mathcal{C}$ ;*
- (iv) *one has  $\tau(\text{NW}(X)) = \text{NW}(X)$  and  $\tau^{-1}(\text{NW}(X)) = \text{NW}(X)$ ;*
- (v) *one has  $\tau(X \setminus \text{NW}(X)) = X \setminus \text{NW}(X)$  and  $\tau^{-1}(X \setminus \text{NW}(X)) = X \setminus \text{NW}(X)$ .*

*Proof.* Let  $C \in \mathcal{C}$  and let  $y \in C$ . We have  $C_\alpha(y) = C$  by Proposition 3.4.(vi). Since  $\tau: X \rightarrow X$  is surjective, there exists  $x \in X$  such that  $\tau(x) = y$ . Let  $C' := C_\alpha(x) \in \mathcal{C}$ . Using Proposition 3.14.(ii), we get

$$\tau(C') = \tau(C_\alpha(x)) \subset C_\alpha(\tau(x)) = C_\alpha(y) = C,$$

which implies  $\rho(C') = C$ . It follows that  $\rho$  is surjective. As  $\mathcal{C}$  is finite,  $\rho$  is bijective. Using Proposition 3.14.(iii), we deduce that  $\rho$  is a poset automorphism of  $\mathcal{C}$ . This shows (i).

To prove (ii), let  $C \in \mathcal{C}$  and let  $C' := \rho^{-1}(C)$ . Since  $\rho(C') = C$ , we have  $\tau(C') \subset C$  and hence  $C' \subset \tau^{-1}(C)$ .

To prove the converse inclusion, suppose that  $x \in \tau^{-1}(C)$ . Then  $y := \tau(x) \in C$ , so that  $C_\alpha(y) = C_\omega(y) = C$  by Proposition 3.4.(vi). It follows from Proposition 3.14.(ii) that

$$\tau(C_\alpha(x)) \subset C_\alpha(\tau(x)) = C_\alpha(y) = C,$$

and

$$\tau(C_\omega(x)) \subset C_\omega(\tau(x)) = C_\omega(y) = C.$$

Therefore, we have  $\rho(C_\alpha(x)) = \rho(C_\omega(x)) = C$ . As  $\rho$  is bijective by (i), this implies  $C_\alpha(x) = C_\omega(x) = \rho^{-1}(C) = C'$ . We deduce that  $x \in C'$  by applying again Proposition 3.4.(vi). This shows that  $\tau^{-1}(C) \subset C'$  and completes the proof of (ii).

To prove (iii), let  $C \in \mathcal{C}$  and let  $C' := \rho(C)$ . We then have  $\tau(C) \subset C'$  by definition of  $\rho$ . As  $\tau^{-1}(C') = \rho^{-1}(C') = C$  by (ii), we conclude that  $\tau(C) = C'$ . This shows (iii).

We have

$$\begin{aligned} \tau^{-1}(\text{NW}(X)) &= \tau^{-1}\left(\bigcup_{C \in \mathcal{C}} C\right) && \text{(by Proposition 3.4.(ii))} \\ &= \bigcup_{C \in \mathcal{C}} \tau^{-1}(C) \\ &= \bigcup_{C \in \mathcal{C}} \rho^{-1}(C) && \text{(by (ii))} \\ &= \bigcup_{C \in \mathcal{C}} C && \text{(since } \rho \text{ permutes } \mathcal{C}\text{)} \\ &= \text{NW}(X) && \text{(by Proposition 3.4.(ii)).} \end{aligned}$$

As  $\tau$  is surjective, we deduce that  $\tau(\text{NW}(X)) = \text{NW}(X)$ . This shows (iv).

Assertion (v) follows from (iv) and the surjectivity of  $\tau$ .  $\square$

**Proposition 3.16.** *Let  $A$  be a finite set, let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type, and let  $\mathcal{C}$  denote the poset of irreducible components of  $X$ . Suppose that  $\tau: X \rightarrow X$  is an injective endomorphism of  $X$  and let  $\rho = \rho_\tau: \mathcal{C} \rightarrow \mathcal{C}$  denote the poset endomorphism of  $\mathcal{C}$  associated with  $\tau$ . Then the following hold:*

- (i) *one has  $\tau(\text{NW}(X)) = \text{NW}(X)$  and  $\tau$  induces by restriction a topological conjugacy of  $\text{NW}(X)$  onto itself;*
- (ii) *the map  $\rho: \mathcal{C} \rightarrow \mathcal{C}$  is a poset automorphism of  $\mathcal{C}$ ;*
- (iii) *for each  $C \in \mathcal{C}$ , one has  $\tau(C) = \rho(C)$  and  $\tau$  induces by restriction a topological conjugacy of  $C$  onto  $\rho(C)$ ;*
- (iv) *one has  $\tau(X \setminus \text{NW}(X)) \subset X \setminus \text{NW}(X)$  and  $\tau^{-1}(X \setminus \text{NW}(X)) = X \setminus \text{NW}(X)$ ;*
- (v)  *$\tau$  is surjective if and only if  $\tau(X \setminus \text{NW}(X)) = X \setminus \text{NW}(X)$*

*Proof.* It follows from Proposition 2.3.(i) that  $\tau(\text{NW}(X)) \subset \text{NW}(X)$ . Thus,  $\tau$  induces, by restriction, an injective endomorphism  $\varphi := \tau|_{\text{NW}(X)}: \text{NW}(X) \rightarrow \text{NW}(X)$ . The subshift  $\text{NW}(X)$  is of finite type by Proposition 3.4.(i) and non-wandering by definition. Therefore  $\text{NW}(X)$  is surjunctive by Corollary 1.4 in [9]. It follows that  $\varphi$  is surjective. We deduce

that  $\varphi: \text{NW}(X) \rightarrow \text{NW}(X)$  is a topological conjugacy from  $\text{NW}(X)$  onto itself. This shows (i).

Observe that the set of irreducible components of  $\text{NW}(X)$  is the same as that of  $X$ , namely  $\mathcal{C}$ . We deduce from (i) that  $\tau$  permutes the elements of  $\mathcal{C}$ . It then follows from Assertion (iii) in Proposition 3.15 that  $\rho$  is a poset automorphism of  $\mathcal{C}$ . Moreover,  $\tau$  induces by restriction a topological conjugacy of  $C$  onto  $\rho(C)$  for every  $C \in \mathcal{C}$ . This shows (ii) and (iii).

Finally, (iv) and (v) follow from (i) and injectivity of  $\tau$ .  $\square$

#### 4. MARKERS AND INJUNCTIVITY

The introduction of markers in symbolic dynamics goes back to Hedlund [14] who used them in his investigation of the automorphism group of full shifts. The technique of markers was extended by Boyle, Lind, and Rudolph [3] and by Kim and Roush [16] to study the automorphism group of topologically mixing subshifts of finite type (see also [17, Chapter 3]). In this section, we use markers to construct endomorphisms of subshifts of finite type having an infinite irreducible component in order to show that these subshifts are not injunctive. For our purpose, we define a marker in the following way.

**Definition 4.1.** Let  $A$  be a set. One says that a pair of words  $(w_0, w_1) \in A^* \times A^*$  is a *marker* if the following conditions are satisfied:

- (M1) one has  $w_0, w_1 \in A^n$  for some  $n \geq 3$ ;
- (M2) one has  $w_0 \neq w_1$ ;
- (M3) one has  $\text{pref}_1(w_0) = \text{suff}_1(w_0) = \text{pref}_1(w_1) = \text{suff}_1(w_1)$ ;
- (M4) for all  $k \in \{2, \dots, n-1\}$  and  $r, s \in \{0, 1\}$ , one has  $\text{suff}_k(w_r) \neq \text{pref}_k(w_s)$ .

Condition (M4) says that, for all  $r, s \in \{0, 1\}$ , the words  $w_r$  and  $w_s$  have only trivial overlaps.

**Example 4.2.** Let  $A := \{0, 1\}$ . Then the pair  $(0100, 0110)$  is a marker while the pair  $(0100, 0010)$  is not a marker.

**Lemma 4.3.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the subshift of finite type associated with  $G$ . Suppose that the subshift  $X$  contains an infinite irreducible component  $C$  and let  $K$  denote the path-connected component of  $G$  associated with  $C$ . Then there exists a marker  $(w_0, w_1) \in V^* \times V^*$  such that  $w_0$  and  $w_1$  are both cycles in  $K$ .*

*Proof.* (cf. [3, Proof of Lemma 2.2]) Since  $C$  is infinite, there exist distinct vertices  $v_0, v_1 \in K$  such that  $v_0 v_1 \in E$  (otherwise, being path-connected,  $K$  would be reduced to a single vertex having a self-loop, and  $C$  would be reduced to a single constant configuration). Let then  $v_1 \cdots v_h v_{h+1}$  be a path in  $K$  going from  $v_1$  to  $v_0 = v_{h+1}$  of minimal length  $h \geq 1$ . It follows that  $\pi := v_0 v_1 \cdots v_h v_0$  is a simple cycle in  $K$  of length  $\ell_\pi = h + 1$ .

From Proposition 3.7, we deduce that there exist  $u, v \in K$  such that  $uv \in (E \cap K^2) \setminus \{v_i v_{i+1} : i = 0, 1, \dots, h\}$ . We claim that there exist  $i \in \{0, 1, \dots, h\}$  and  $w \in K \setminus \{v_{i+1}\}$  such that  $v_i w \in E$ . Indeed, if  $u = v_i$  for some  $i \in \{0, 1, \dots, h\}$ , then we can take  $w := v$ . Otherwise, let  $v'_0 v'_1 \cdots v'_t$  be a path in  $K$  of minimal length  $t \geq 1$  connecting  $\pi$  to  $u$ . Thus,

there exists  $i \in \{0, 1, \dots, h\}$  such that  $v'_0 = v_i$ . By minimality, we have  $v'_1 \neq v_{i+1}$ . One may then take  $w := v'_1$ . This proves the claim. Up to relabeling the indices, we may assume that  $i = 0$ .

Suppose first that  $w \neq v_0$ . Let  $u_1 u_2 \cdots u_t v_{t'}$ ,  $0 \leq t' \leq h$ , be a path in  $K$  of minimal length  $t \geq 0$  connecting  $w = u_1$  to  $\pi$  (note that  $t = 0$  corresponds to the case  $u_1 = v_{t'}$  for some  $t' \in \{2, \dots, h\}$ ). Then  $\zeta := v_0 u_1 u_2 \cdots u_t v_{t'} v_{t'+1} \cdots v_h v_0$  is a simple cycle based at  $v_0$  of length  $\ell_\zeta = t + h - t' + 2$ . We then set

$$w_0 := \pi \# \pi \# \zeta \# \pi \# \zeta \# \zeta \quad \text{and} \quad w_1 := \pi \# \pi \# \zeta \# \zeta \# \pi \# \zeta.$$

Observe that the words  $w_0, w_1 \in V^*$  are such that

$$|w_0| = |w_1| = n := 3(\ell_\pi + \ell_\zeta) + 1 = 6h + 3(t - t') + 10 \geq 3,$$

$w_0 \neq w_1$ , and  $\text{pref}_1(w_0) = \text{suff}_1(w_0) = \text{pref}_1(w_1) = \text{suff}_1(w_1) = v_0$ . It follows that  $(w_0, w_1)$  satisfies Conditions (M1), (M2), and (M3) in Definition 4.1.

Suppose now that  $w = v_0$ . This means that  $\theta := v_0 w = v_0 v_0$  is a self-loop at  $v_0$  in  $K$ , of length  $\ell_\theta = 1$ . We then set

$$w_0 := \pi \# \pi \# \theta \# \pi \# \theta \# \theta \quad \text{and} \quad w_1 := \pi \# \pi \# \theta \# \theta \# \pi \# \theta.$$

Observe that the words  $w_0, w_1 \in V^*$  are such that

$$|w_0| = |w_1| = n := 3(\ell_\pi + \ell_\theta) + 1 = 3h + 7 \geq 3,$$

$w_0 \neq w_1$ , and  $\text{pref}_1(w_0) = \text{suff}_1(w_0) = \text{pref}_1(w_1) = \text{suff}_1(w_1) = v_0$ . It follows that  $(w_0, w_1)$  satisfies Conditions (M1), (M2), and (M3) in Definition 4.1.

We claim that, in either case,  $(w_0, w_1)$  also satisfies condition (M4) in Definition 4.1. First note that, by minimality, the vertex  $v_0$  only occurs at the beginning and at the end of the word  $\pi$  (resp.  $\zeta$ , resp.  $\theta$ ). Consequently,  $v_0$  occurs in  $w_0$  and  $w_1$  exactly 7 times, say at positions  $1 = i_1 < i_2 < \cdots < i_7 = n$  and  $1 = j_1 < j_2 < \cdots < j_7 = n$ , respectively. Thus, when comparing  $\text{suff}_k(w_r)$  and  $\text{pref}_k(w_s)$  for  $k \in \{2, \dots, n-1\}$  and  $r, s \in \{0, 1\}$ , we may limit ourselves to the cases corresponding to  $k = n - i_m$  (resp.  $k = n - j_m$ ) for  $m = 2, 3, \dots, 6$ .

In order to prove the claim, it thus suffices to look at the following tables, where in the first line we have factorized the word  $w_r$  into blocks (made up of  $\pi$ s,  $\zeta$ s, or  $\theta$ s) and in the subsequent five lines we have factorized the blocks (also made up of  $\pi$ s,  $\zeta$ s, or  $\theta$ s) for the five meaningful prefixes of  $w_s$ . A comparison of the columns shows that there is always a mismatch for some position of the corresponding blocks.

This proves the claim.

We deduce that  $(w_0, w_1)$  is a marker. □

**Lemma 4.4.** *Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the subshift of finite type associated with  $G$ . Suppose that there exists a marker  $(w_0, w_1) \in V^* \times V^*$  such that  $w_0$  and  $w_1$  are both cycles in  $G$ . Then the subshift  $X$  is not injunctive.*

*Proof.* Denote by  $\ell$  the common length of the cycles  $w_0$  and  $w_1$  (so that  $\ell = |w_0| - 1 = |w_1| - 1$ ).

$\pi$	$\pi$	$\zeta$	$\pi$	$\zeta$	$\zeta$
	$\pi$	$\pi$	$\zeta$	$\pi$	$\zeta$
		$\pi$	$\pi$	$\zeta$	$\pi$
			$\pi$	$\pi$	$\zeta$
				$\pi$	$\pi$
					$\pi$

TABLE 1.  $w_0$  has no nontrivial overlaps with itself:  $\text{suff}_k(w_0) \neq \text{pref}_k(w_0)$  for  $k \neq 1, n$ .

$\pi$	$\pi$	$\zeta$	$\pi$	$\zeta$	$\zeta$
	$\pi$	$\pi$	$\zeta$	$\zeta$	$\pi$
		$\pi$	$\pi$	$\zeta$	$\zeta$
			$\pi$	$\pi$	$\zeta$
				$\pi$	$\pi$
					$\pi$

TABLE 2.  $w_0$  has no nontrivial overlaps with  $w_1$ :  $\text{suff}_k(w_0) \neq \text{pref}_k(w_1)$  for  $k \neq 1, n$ . This covers the case  $w \neq v_0$ . By replacing  $\zeta$  by  $\theta$ , the case  $w = v_0$  is covered as well.

$\pi$	$\pi$	$\zeta$	$\zeta$	$\pi$	$\zeta$
	$\pi$	$\pi$	$\zeta$	$\zeta$	$\pi$
		$\pi$	$\pi$	$\zeta$	$\zeta$
			$\pi$	$\pi$	$\zeta$
				$\pi$	$\pi$
					$\pi$

TABLE 3.  $w_1$  has no nontrivial overlaps with itself:  $\text{suff}_k(w_1) \neq \text{pref}_k(w_1)$  for  $k \neq 1, n$ . This covers the case  $w \neq v_0$ . By replacing  $\zeta$  by  $\theta$ , the case  $w = v_0$  is covered as well.

Given a configuration  $x \in X$ , define the configuration  $\tau(x)$  as follows. For  $i \in \mathbb{Z}$ , if there exist  $r, s \in \{0, 1\}$  such that  $x_{[i, i+2\ell]} = w_r \# w_s$ , define  $\tau(x)_{[i, i+\ell]} := w_t$ , where  $t := r + s \pmod 2$ . All other entries of  $x$  remain unchanged. The words  $w_0$  and  $w_1$  have only trivial overlaps by (M4). Moreover, by (M3), there is a vertex  $v \in V$  such that  $w_0$  and  $w_1$  are

both cycles based at  $v$ . It follows that the configuration  $\tau(x)$  is well defined and that one has  $\tau(x) \in X$ .

The map  $\tau: X \rightarrow X$  is an endomorphism of  $X$  since it clearly satisfies condition (b) in Proposition 2.5 for  $m = 2\ell + 1$ .

Consider now the periodic configurations  $x^{(0)} \in X$  (resp.  $x^{(1)} \in X$ ) associated with the cycle  $w_0$  (resp.  $w_1$ ). We have  $x^{(0)} \neq x^{(1)}$  since  $w_0 \neq w_1$  by (M2). As  $\tau(x^{(0)}) = \tau(x^{(1)}) = x^{(0)}$ , it follows that  $\tau$  is not injective.

Let us show that  $\tau$  is surjective. Let  $y \in X$ . If there are no occurrences in  $y$  of any of the words  $w_r \# w_s$ ,  $r, s \in \{0, 1\}$ , then  $y = \tau(y)$ . Otherwise, consider the set  $S \subset \mathbb{Z}$  consisting of all  $i \in \mathbb{Z}$  such that there exists  $j \in \mathbb{Z}$  such that  $j \leq i \leq j + 2\ell$  and  $x_{[j, j+2\ell]} = w_r \# w_s$  for some  $r, s \in \{0, 1\}$ . We construct  $x \in X$  such that  $\tau(x) = y$  as follows. If  $i \in \mathbb{Z} \setminus S$ , we set  $x_i := y_i$ . Suppose now  $i \in S$ . Let  $I$  be the maximal interval of  $\mathbb{Z}$  such that  $i \in I \subset S$ . The interval  $I$  is of one of the following forms:

- (1)  $I = [a, a + k\ell]$  for some  $a, k \in \mathbb{Z}$  with  $k \geq 2$ ;
- (2)  $I = [a, \infty)$  for some  $a \in \mathbb{Z}$ ;
- (3)  $I = (-\infty, a]$  for some  $a \in \mathbb{Z}$ ;
- (4)  $I = \mathbb{Z}$ .

When  $I$  is of the first form, we have  $y_I = w_{r_1} \# \cdots \# w_{r_k}$ , where  $r_j \in \{0, 1\}$  for all  $j \in \{1, \dots, k\}$ . We then define  $x_I$  by replacing  $y_I$  by  $w_{s_1} \# \cdots \# w_{s_k}$ , where  $s_k := r_k$  and  $s_j := r_j + r_{j+1} \pmod 2$  for all  $j \in \{1, \dots, k-1\}$ .

When  $I$  is of the second form, we have  $y_I = w_{r_0} \# w_{r_1} \# w_{r_2} \# \cdots$ , where  $r_j \in \{0, 1\}$  for all  $j \in \mathbb{N}$ . We then define  $x_I$  by replacing  $y_I$  by  $w_{s_0} \# w_{s_1} \# w_{s_2} \# \cdots$ , where  $s_0 := r_0$  and  $s_j := s_{j-1} + r_j \pmod 2$  for all  $j \in \{1, 2, \dots\}$ .

When  $I$  is of the third form, we have  $y_I = \cdots \# w_{r_{-2}} \# w_{r_{-1}} \# w_{r_0}$ , where  $r_{-j} \in \{0, 1\}$  for all  $j \in \mathbb{N}$ . We then define  $x_I$  by replacing  $y_I$  by  $\cdots \# w_{s_{-2}} \# w_{s_{-1}} \# w_{s_0}$ , where  $s_0 := r_0$  and  $s_j := s_{j+1} + r_j \pmod 2$  for all  $j \in \{-1, -2, \dots\}$ .

Finally, when  $y$  is of the fourth form, we have  $S = \mathbb{Z}$  and  $y = \cdots \# w_{r_{-1}} \# w_{r_0} \# w_{r_1} \# \cdots$  (the precise position of  $y_0$  is irrelevant) and in this case we take  $x := \cdots \# w_{s_{-1}} \# w_{s_0} \# w_{s_1} \# \cdots$ , where  $s_0 := r_0$ ,  $s_j := s_{j-1} + r_j \pmod 2$  for all  $j \in \{1, 2, \dots\}$ , and  $s_j := s_{j+1} + r_j \pmod 2$  for all  $j \in \{-1, -2, \dots\}$ .

We then clearly have  $x \in X$  and  $\tau(x) = y$ . This shows that  $\tau$  is surjective. As  $\tau$  is not injective, we conclude that  $X$  is not injective.  $\square$

**Proposition 4.5.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Suppose that  $X$  contains an infinite irreducible component. Then the subshift  $X$  is not injective.*

*Proof.* As we can assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ , this immediately follows from Lemma 4.3 and Lemma 4.4.  $\square$

## 5. CONSTRUCTION OF $\tau_C$

Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the subshift of finite type associated with  $G$ . Fix some irreducible component  $C$  of  $X$ . In this section, we

construct an injective endomorphism  $\tau_C: X \rightarrow X$ . The fact that it may fail to be surjective will be used in Section 7.2 for proving properties of surjunctive subshifts of finite type stated in Theorem 1.1. The construction of  $\tau_C$  and the proof that it is an injective endomorphism are divided into several steps.

**Step 1.** Denote by  $R$  the set of recurrent vertices of  $G$ . Then there exists an integer  $\ell = \ell(G) \geq 1$  such that, for each  $v \in R$ , there is a cycle of length  $\ell$  based at  $v$  in  $G$ . Indeed, by definition of recurrence, for each  $v \in R$ , we can find a cycle based at  $v$ . Denoting by  $\ell_v$  the length of such a cycle, we can take  $\ell := \text{lcm}\{\ell_v : v \in R\}$ . In the sequel, we fix for each  $v \in R$  a *preferred* cycle  $\pi_v$  of length  $\ell$  based at  $v$  in  $G$ . Note that if  $K$  is the unique path-component of  $G$  containing  $v$ , then all the vertices of  $\pi_v$  lie in  $K$ .

**Step 2.** Fix some irreducible component  $C$  of  $X$ . Denote by  $\mathcal{K}$  the poset of path-components of  $G$ . Let  $K \in \mathcal{K}$  denote the path-component of  $G$  associated with  $C$ . Set  $\mathcal{K}_1 := \{K' \in \mathcal{K} : K' \prec K\}$  and  $\mathcal{K}_2 := \mathcal{K} \setminus \mathcal{K}_1$ . Note that if  $K' \in \mathcal{K}_1$  (resp.  $K' \in \mathcal{K}_2$ ) and  $K'' \in \mathcal{K}$  satisfy  $K'' \preceq K'$  (resp.  $K' \preceq K''$ ) then we have  $K'' \in \mathcal{K}_1$  (resp.  $K'' \in \mathcal{K}_2$ ). Denote by  $V_1 \subset V$  the set of vertices  $v$  such that  $v \prec u$  for some (equivalently, all)  $u \in K$  and set  $V_2 := V \setminus V_1$ . Also set  $R_1 := R \cap V_1$  and  $R_2 := R \cap V_2$ . Observe that  $R_1 = \bigcup_{K' \in \mathcal{K}_1} K'$  and  $R_2 = \bigcup_{K' \in \mathcal{K}_2} K'$ .

**Step 3.** We are now in a position to define the map  $\tau = \tau_C: X \rightarrow X$  we alluded to above. Let  $x \in X$  and let  $\Gamma = \{K_0, K_1, \dots, K_n\}$ , where  $n \geq 0$  and  $K_0 \prec K_1 \prec \dots \prec K_n$ , denote the chain of path-components of  $x$ . Then we are in one and only one of the following three cases:

- Case 1: The terminal path-component  $K_n$  of  $x$  is in  $\mathcal{K}_1$ . Note that under this assumption, all path-components  $K_j$ s of  $x$  lie in  $\mathcal{K}_1$  as well. We then set  $\tau(x) := \sigma^\ell(x)$ , where the integer  $\ell = \ell(G) \geq 1$  is given by Step 1.
- Case 2: The initial path-component  $K_0$  of  $x$  is in  $\mathcal{K}_2$ . Note that under this assumption, all path-components  $K_j$ s of  $x$  lie in  $\mathcal{K}_2$  as well. We then set  $\tau(x) := \sigma^{-\ell}(x)$ .
- Case 3: The initial path-component  $K_0$  of  $x$  is in  $\mathcal{K}_1$  and its terminal path-component  $K_n$  is in  $\mathcal{K}_2$ . Then, with the notation in (3.1), we have  $n = n(x) \geq 1$  and there exists a unique integer  $h = h(x) \in \{0, 1, \dots, n-1\}$  such that  $K_j \in \mathcal{K}_1$  for all  $0 \leq j \leq h$  and  $K_j \in \mathcal{K}_2$  for all  $h+1 \leq j \leq n$ .

Let now  $u := x_{b_h} \in R_1$  (resp.  $v := x_{a_{h+1}} \in R_2$ ) and let  $\pi_u = c_0 \cdots c_\ell$  (resp.  $\pi_v = c'_0 \cdots c'_\ell$ ) denote the preferred cycle of length  $\ell$  based at  $u$  (resp.  $v$ ). As remarked above, we have  $c_0, \dots, c_\ell \in K_h$  (resp.  $c'_0, \dots, c'_\ell \in K_{h+1}$ ). We then set,

for every  $i \in \mathbb{Z}$ ,

$$(5.1) \quad \tau(x)_i := \begin{cases} x_{i+\ell} & \text{if } i \in (-\infty, b_h - \ell], \\ c_{i-b_h+\ell} & \text{if } i \in [b_h - \ell, b_h - 1], \\ x_i & \text{if } i \in [b_h, a_{h+1}], \\ c'_{i-a_{h+1}} & \text{if } i \in [a_{h+1}, a_{h+1} + \ell - 1], \\ x_{i-\ell} & \text{if } i \in [a_{h+1} + \ell, \infty). \end{cases}$$

**Remark 5.1.** In either case, the chain of path-components of  $\tau(x)$  is the same as that of  $x$ . Observe that in Case 1 (resp. Case 2), we have  $x_i \in V_1$  (resp.  $x_i \in V_2$ ) for all  $i \in \mathbb{Z}$ . In Case 3, we have  $h(\tau(x)) = h(x)$ ,  $b_h(\tau(x)) = b_h(x)$ , and  $a_{h+1}(\tau(x)) = a_{h+1}(x)$ . In Case 3, all vertices  $x_i$  which are in  $V_1$  (resp. in  $V_2 \setminus [b_h + 1, a_{h+1}]$ ) are shifted to the left (resp. to the right) by  $\ell$  positions in  $\tau(x)$ .

It is clear that  $\tau(x) \in X$  for all  $x \in X$  (in Cases 1 and 2, the configuration  $\tau(x)$  is obtained from  $x$  by shifting it while, in Case 3, the configuration  $\tau(x)$  is obtained from  $x$  by inserting the two cycles  $\pi_u$  and  $\pi_v$  based at vertices  $u$  and  $v$ , respectively).

**Step 4.** Let us show that the map  $\tau: X \rightarrow X$  is injective. Let  $x, y \in X$  such that  $y = \tau(x)$ . As observed above, the configurations  $x$  and  $y$  have the same initial path-component  $K_0 := K_0(x) = K_0(y)$  and the same terminal path-component  $K_n := K_n(x) = K_n(y)$ , where  $n := n(x) = n(y)$ . We distinguish the following mutually exclusive cases.

Case 1:  $K_n \in \mathcal{K}_1$ . Then, we have  $y = \sigma^\ell(x)$ , so that  $x = \sigma^{-\ell}(y)$ .

Case 2:  $K_0 \in \mathcal{K}_2$ . Then, we have  $y = \sigma^{-\ell}(x)$ , so that  $x = \sigma^\ell(y)$ .

Case 3:  $K_0 \in \mathcal{K}_1$  and  $K_n \in \mathcal{K}_2$ . Then, we deduce from (5.1) that

$$x_i = \begin{cases} y_{i-\ell} & \text{if } i \in (-\infty, b_h], \\ y_i & \text{if } i \in [b_h + 1, a_{h+1}], \\ y_{i+\ell} & \text{if } i \in [a_{h+1} + 1, +\infty). \end{cases}$$

In each case, the configuration  $x$  is uniquely determined by the configuration  $y$ . It follows that  $\tau$  is injective.

**Step 5.** Let us show that  $\tau$  commutes with the shift map  $\sigma$ . Let indeed  $x \in X$ . Observe that the chain of path-components of  $\sigma(x)$  is the same as that of  $x$ . In particular,  $x$  and  $\sigma(x)$  have the same initial path-component  $K_0 := K_0(x) = K_0(\sigma(x))$  and the same terminal path-component  $K_n := K_n(x) = K_n(\sigma(x))$ , where  $n := n(x) = n(\sigma(x))$ .

We distinguish the same mutually exclusive cases as above.

Case 1:  $K_n \in \mathcal{K}_1$ . Then  $\tau(x) = \sigma^\ell(x)$  and  $\tau(\sigma(x)) = \sigma^\ell(\sigma(x)) = \sigma^{\ell+1}(x)$ . We thus have  $(\sigma \circ \tau)(x) = \sigma(\sigma^\ell(x)) = \sigma^{\ell+1}(x) = \tau(\sigma(x)) = (\tau \circ \sigma)(x)$ .

Case 2:  $K_0 \in \mathcal{K}_2$ . Then  $\tau(x) = \sigma^{-\ell}(x)$  and  $\tau(\sigma(x)) = \sigma^{-\ell}(\sigma(x)) = \sigma^{-\ell+1}(x)$ . We thus have  $(\sigma \circ \tau)(x) = \sigma(\sigma^{-\ell}(x)) = \sigma^{-\ell+1}(x) = \tau(\sigma(x)) = (\tau \circ \sigma)(x)$ .

Case 3:  $K_0 \in \mathcal{K}_1$  and  $K_n \in \mathcal{K}_2$ . We then have  $h(\sigma(x)) = h(x)$ ,  $a_j(\sigma(x)) = a_j(x) - 1$  for all  $j = 1, 2, \dots, n$ , and  $b_j(\sigma(x)) = b_j(x) - 1$  for all  $j = 0, 1, \dots, n-1$ . From (5.1), we deduce that  $(\tau \circ \sigma)(x) = \tau(\sigma(x)) = \sigma(\tau(x)) = (\sigma \circ \tau)(x)$ .

This shows that  $(\tau \circ \sigma)(x) = (\sigma \circ \tau)(x)$  for all  $x \in X$ . Therefore, we have  $\tau \circ \sigma = \sigma \circ \tau$ .

**Step 6.** Let  $x \in X$  and  $y := \tau(x)$ . We claim that  $y_0$  only depends on the word  $x_{[-\ell, \ell]} \in V^{2\ell+1}$ . We distinguish the following three cases.

(1)  $x_\ell \in V_1$ . Then, we have (cf. Remark 5.1)

$$y_0 = \sigma^\ell(x)_0 = x_\ell.$$

(2)  $x_{-\ell} \in V_2$ . Then, we have (cf. Remark 5.1):

$$y_0 = \sigma^{-\ell}(x)_0 = x_{-\ell}.$$

(3)  $x_{-\ell} \in V_1$  and  $x_\ell \in V_2$ . We are then in Case 3. Let  $\Gamma(x) = \{K_0, K_1, \dots, K_n\}$ , with  $K_0 \prec K_1 \prec \dots \prec K_n$ , denote the chain of path-components of  $x$  and recall that  $h \in \{0, 1, \dots, n-1\}$  is the unique index such that  $K_h \in \mathcal{K}_1$  and  $K_{h+1} \in \mathcal{K}_2$ . Recall also that  $b_h$  (resp.  $a_{h+1}$ ) is the largest (resp. least) integer such that  $x_{b_h} \in K_h$  (resp.  $x_{a_{h+1}} \in K_{h+1}$ ). Thus, we have  $b_h < \ell$  and  $a_{h+1} > -\ell$  in this case.

We distinguish the following subcases:

(3.1):  $0 \leq b_h$ . We have

$$y_{[-\ell, b_h]} = x_0 \cdots x_{b_h} c_1 c_2 \cdots c_\ell.$$

Observe that the values  $c_1, c_2, \dots, c_\ell$  are uniquely determined by  $x_{b_h}$ . As  $-\ell \leq 0 \leq b_h < \ell$ , the claim follows in this case.

(3.2):  $b_h \leq 0$  and  $a_{h+1} \leq 0$ . We have

$$y_{[a_{h+1}, 2\ell]} = x_{a_{h+1}} c'_1 c'_2 \cdots c'_\ell x_{a_{h+1}+1} \cdots x_\ell.$$

Observe that the values  $c'_1, c'_2, \dots, c'_\ell$  are uniquely determined by  $x_{a_{h+1}}$ . As  $-\ell < a_{h+1} \leq 0 \leq 2\ell$ , the claim follows in this case.

(3.3):  $b_h \leq 0$  and  $0 \leq a_{h+1} \leq \ell$ . We have

$$y_{[b_h, a_{h+1}]} = x_{b_h} \cdots x_{a_{h+1}}.$$

As  $-\ell < b_h \leq 0 \leq a_{h+1} \leq \ell$ , the claim follows in this case.

(3.4):  $b_h \leq 0$  and  $\ell < a_{h+1}$ . We have

$$y_{[b_h, \ell]} = x_{b_h} \cdots x_\ell.$$

As  $-\ell < b_h \leq 0 \leq \ell$ , the claim follows in this case as well.

This completes the proof that  $\tau(x)_0$  only depends on the word  $x_{[-\ell, \ell]}$ .

**Step 7.** It follows from the previous step that there is a map  $\mu: V^{2\ell+1} \rightarrow V$  such that  $\tau(x)_0 = \mu(x_{[-\ell, \ell]})$  for all  $x \in X$ . As  $\tau$  commutes with the shift (cf. Step 5), we deduce from Proposition 2.5 that  $\tau: X \rightarrow X$  is an endomorphism of  $X$ .

## 6. CONSTRUCTION OF $\tau^C$

Let  $G = (V, E)$  be a finite directed graph and let  $X \subset V^{\mathbb{Z}}$  denote the subshift of finite type associated with  $G$ . Let  $C$  be an infinite irreducible component of  $X$  which is minimal in the poset of all irreducible components of  $X$ . In this section, we construct a surjective endomorphism  $\tau^C: X \rightarrow X$ . The fact that it may fail to be pre-injective will be used in Section 7.2 for proving the properties of Moore subshifts of finite type stated in Theorem 1.1.

We first need some auxiliary lemmas of a combinatorial nature.

**Lemma 6.1.** *Let  $K = (V, E)$  be a finite strongly connected directed graph. Suppose that the associated subshift of finite type  $X \subset V^{\mathbb{Z}}$  is infinite. Then there exists an integer  $n = n(K) \geq 1$  such that for each  $v \in V$  there are two distinct cycles of length  $n$  based at  $v$  in  $K$ .*

*Proof.* Fix  $v_0 \in V$ . Then there exists  $v_1 \in K \setminus \{v_0\}$  such that  $v_0v_1 \in E$  (otherwise, being path-connected,  $K$  would be reduced to the single vertex  $v_0$  with a self-loop, and  $X$  would consist only of the configuration with constant value  $v_0$ , contradicting the hypothesis that  $X$  is infinite). Let then  $v_1 \cdots v_hv_0$  be a path in  $K$  going from  $v_1$  to  $v_0$  of minimal length  $h \geq 1$ . As in the proof of Lemma 4.3, we can find  $i \in \{0, 1, \dots, h\}$  and  $w \in K \setminus \{v_{i+1}\}$  such that  $v_iw \in E$ . As  $K$  is path-connected, we can find a path  $w_0 \cdots w_k$  in  $K$  going from  $w = w_0$  to  $v_0 = w_k$  of minimal length  $k \geq 1$ . Then,  $v_0v_1 \cdots v_hv_0$  and  $v_0v_1 \cdots v_iw_0w_1 \cdots w_{k-1}v_0$  are distinct cycles in  $K$  and  $(v_0v_1 \cdots v_hv_0)^{s\sharp} \neq (v_0 \cdots v_iw_0w_1 \cdots w_{k-1}v_0)^{t\sharp}$  for all integers  $s, t \geq 1$ . In particular, taking  $s := i + k + 1$  and  $t := h + 1$  yields two distinct cycles based at  $v_0$  of the same length  $n_{v_0} = (h + 1)(i + k + 1)$ .

One may then take  $n := \text{lcm}(n_v : v \in V)$ . □

**Remark 6.2.** We may also deduce Lemma 6.1 by using results from the Perron-Frobenius theory of non-negative matrices (cf. [18, Chapter 4], [17, Section 1.3], [8, Chapter 6]) in the following way.

The *adjacency matrix* of  $K = (V, E)$  is the matrix  $A = (A_{uv})_{u, v \in V}$  where  $A_{uv} = 1$  if  $uv \in E$  and  $A_{uv} = 0$  otherwise. For every integer  $m \geq 1$  and any  $u, v \in V$ , the  $uv$ -entry  $(A^m)_{uv}$  of the matrix  $A^m$  is the number of paths of length  $m$  in  $K$  going from  $u$  to  $v$ . Since  $K$  is strongly connected, the matrix  $A$  is irreducible (i.e., for all  $u, v \in V$ , there exists an integer  $m = m(u, v) \geq 1$  such that  $(A^m)_{uv} \geq 1$ ). The entropy of the vertex subshift  $X$  associated with  $K$  is given by  $h(X) = \log \lambda$ , where  $\lambda$  is the Perron-Frobenius eigenvalue of  $A$  (cf. [18, Theorem 4.3.1 and Theorem 4.3.3]). Since  $X$  is infinite, we have  $\lambda > 1$  (cf. [18, Exercise 4.3.1 and Corollary 4.4.9]). The *period* of  $A$  is the integer  $p \geq 1$  defined by  $p := \text{gcd}\{n \geq 1 : (A^n)_{vv} > 0\}$ , where  $v \in V$  (irreducibility of  $A$  implies that  $p$  is finite and does not depend on the choice of  $v$ ). If  $p = 1$ , the matrix  $A$  is primitive (i.e., there exists an integer  $m \geq 1$  such that all entries of  $A^m$  are positive).

There exists a partition  $V = V_1 \sqcup \cdots \sqcup V_p$  and primitive irreducible matrices  $B_k = ((B_k)_{uv})_{u,v \in V_k}$ ,  $k = 1, \dots, p$ , each having Perron-Frobenius eigenvalue  $\lambda^p > 1$ , such that, up to reindexing, the matrix  $A^p$  admits the diagonal block decomposition

$$A^p = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_p \end{pmatrix}.$$

It follows from the Perron-Frobenius theory for primitive matrices that, for all  $u, v \in V_k$  and  $k = 1, \dots, p$ , the sequence  $(B_k^m)_{uv}/\lambda^m$  converges to a finite positive limit as  $m \rightarrow \infty$  (cf. [18, Theorem 4.5.12], [17, Theorem 1.3.2(f)]). Since  $\lambda > 1$ , we deduce that there exists an integer  $m \geq 1$  such that  $(B_k^m)_{uv} \geq 2$  for all  $u, v \in V_k$  and  $k = 1, \dots, p$ . Now, for every  $v \in V$ , the entry  $(A^{pm})_{vv}$  belongs to one of the diagonal blocks  $B_k^m$ . Hence  $(A^{pm})_{vv} \geq 2$  for all  $v \in V$ , so that we can take  $n := pm$ .

**Lemma 6.3.** *Let  $K = (V, E)$  be a finite directed graph and let  $n$  be a positive integer. Then there exist positive integers  $\ell = \ell(K, n)$  and  $N = N(K, n)$  such that, for every path  $v_0 v_1 \cdots v_N$  of length  $N$  in  $K$ , there exist integers  $t, d \geq 1$  and  $0 \leq i_1 < \cdots < i_t \leq N$  such that*

- $i_j + dn \leq i_{j+1}$  for  $j = 1, \dots, t-1$ ;
- $v_{i_j} = v_{i_j+dn}$  for  $j = 1, \dots, t$ ;
- $tdn = \ell$ .

*Proof.* Let  $m := |V|$  and set  $M := mn$  and  $\ell := \text{lcm}(n, 2n, \dots, Mn)$ . Note that  $\ell$  is a multiple of  $n$  and that  $\ell/(dn)$  is a positive integer for all  $d = 1, 2, \dots, M$ . Define  $B := 1 + \sum_{d=1}^M (\ell/(dn) - 1)$  and set  $N := BM$ .

Let  $v_0 v_1 \cdots v_N$  be a path of length  $N$  in  $K$ . Consider the  $B$  consecutive subpaths  $\pi_r := v_{rM} v_{rM+1} \cdots v_{rM+M}$ ,  $r = 0, \dots, B-1$ , of length  $M$ . For each  $r = 0, \dots, B-1$  consider the  $M+1$  pairs  $(v_{rM+a}, rM+a \bmod n) \in V \times \mathbb{Z}/n\mathbb{Z}$ ,  $a = 0, \dots, M$ . Since  $|V \times \mathbb{Z}/n\mathbb{Z}| = mn = M$ , it follows from the pigeonhole principle that there exist indices  $rM \leq k_r < k'_r \leq rM+M$  such that  $v_{k_r} = v_{k'_r}$  and  $k_r \equiv k'_r \pmod{n}$ . Therefore  $\ell_r := k'_r - k_r$  is a positive multiple of  $n$ . Write  $\ell_r = d_r n$ , where  $1 \leq d_r \leq M$ .

We claim that there exists some  $d \in \{1, \dots, M\}$  such that  $d_r = d$  for at least  $t := \ell/(dn)$  values of  $r$ . Indeed, assume, for contradiction, that for every  $d \in \{1, \dots, M\}$  the value  $d$  occurs strictly fewer than  $t$  times among the  $d_r$ 's. Since the number of occurrences is an integer, this means that each  $d$  occurs at most  $t-1$  times. Summing over all possible values of  $d$ , we obtain the bound  $B \leq \sum_{d=1}^M (\ell/(dn) - 1)$ , a contradiction with the definition of  $B$ . This proves the claim.

Let then  $r_1 < r_2 < \cdots < r_t$  be values of  $r$  such that  $d_{r_j} = d$ , and set  $i_j := k_{r_j}$  for  $j = 1, \dots, t$ . Then  $0 \leq i_1 < \cdots < i_t \leq N$  and  $tdn = \ell$ . Moreover, since the chosen repetitions lie in the distinct subpaths  $\pi_{r_j}$ ,  $j = 1, \dots, t$ , we have  $i_j + dn = k_{r_j} + dn = k'_{r_j} \leq r_j M + M \leq r_{j+1} M \leq k_{r_{j+1}} = i_{j+1}$  for  $j = 1, \dots, t-1$ . Finally, by construction,  $v_{i_j} = v_{k_{r_j}} = v_{k'_{r_j}} = v_{k'_{r_j} + dn} = v_{i_j + dn}$  for all  $j = 1, \dots, t$ .  $\square$

**Remark 6.4.** For each path  $v_0v_1 \cdots v_N$  of length  $N$  in  $K$ , let  $\mathcal{A}(v_0v_1 \cdots v_N)$  denote the finite set of all pairs  $(t, (i_j)_{j=1}^t)$  satisfying the conclusions of Lemma 6.3. By the lemma, this set is nonempty. We choose one such pair canonically as follows. First, choose  $t$  minimal among all admissible pairs. Among the pairs with this value of  $t$ , choose  $(i_1, \dots, i_t)$  maximal with respect to the right lexicographic order, that is, comparing first  $i_t$ , then  $i_{t-1}$ , and so on. Since the set of admissible pairs is finite and the right lexicographic order is total, each step of the selection procedure determines a unique maximal element. Hence the above procedure selects a unique admissible pair, which we denote by  $T(v_0v_1 \cdots v_N) \in \mathcal{A}(v_0v_1 \cdots v_N)$ .

For example, let  $u_0 \in K$  and suppose that  $u_0u_1 \cdots u_n$  is a cycle of length  $n$  based at  $u_0$  in  $K$ . Consider a path  $v_0v_1 \cdots v_N$  of length  $N$  in  $K$  such that  $v_{N-\ell} \cdots v_N = (u_0u_1 \cdots u_n)^{\#}$ , where  $\ell = dn$ . Then  $t = 1$  and  $i_1 = N - \ell$ , so that  $T(v_0v_1 \cdots v_N) = (1, N - \ell)$ .

We are now in a position to introduce the map  $\tau^C: X \rightarrow X$  we alluded to above. Its construction and the proof that it is a surjective endomorphism are divided into several steps.

**Step 1.** Let  $K$  denote the path-component of  $G$  associated with  $C$ .

Let  $n = n(K)$  be as in Lemma 6.1 and let  $N = N(K, n)$  and  $\ell = \ell(K, n)$  be the positive integers provided by Lemma 6.3.

Define a map  $\tau = \tau^C: X \rightarrow X$  as follows. Let  $x \in X$  and let  $\{K_0, K_1, \dots, K_m\}$ , where  $K_0 \prec K_1 \prec \cdots \prec K_m$ , denote the chain of path-components of  $x$ . If the initial path-component  $K_0$  is different from  $K$ , we set  $\tau(x) := \sigma^\ell(x)$ . If the terminal path-component  $K_m$  is equal to  $K$ , i.e., if  $x \in C$ , we set  $\tau(x) := x$ . Otherwise, there exists a unique  $b \in \mathbb{Z}$  such that  $x_i \in K$  for all  $i \in (-\infty, b]$  and  $x_j \in V \setminus K$  for all  $j \in [b + 1, \infty)$ . Consider the admissible pair  $T(x) := T(x_{b-N}x_{b-N+1} \cdots x_b) = (t, (i_j)_{j=1}^t) \in \mathcal{A}(x_{b-N}x_{b-N+1} \cdots x_b)$  defined in Remark 6.4.

We then set, for every  $i \in \mathbb{Z}$ ,

$$(6.1) \quad \tau(x)_i := \begin{cases} x_i & \text{if } i \in (-\infty, i_1], \\ x_{i+dn} & \text{if } i \in [i_1, i_2 - dn], \\ x_{i+2dn} & \text{if } i \in [i_2 - dn, i_3 - 2dn], \\ \dots & \\ x_{i+(t-1)dn} & \text{if } i \in [i_{t-1} - (t-2)dn, i_t - (t-1)dn], \\ x_{i+\ell} & \text{if } i \in [i_t - (t-1), +\infty). \end{cases}$$

In other words,  $\tau$  squeezes the cycle  $x_{i_j}x_{i_j+1} \cdots x_{i_j+dn}$  (by collapsing  $x_{i_j+1}, x_{i_j+2}, \dots, x_{i_j+dn}$  to  $x_{i_j}$ , on the left) for  $j = 1, 2, \dots, t$ .

By uniqueness of the admissible pair  $T(x)$  (cf. Remark 6.4), the map  $\tau$  is well defined.

**Step 2.** Let us show that the map  $\tau: X \rightarrow X$  commutes with the shift map  $\sigma$ . Let indeed  $x \in X$ . Observe that the chain of path-components of  $\sigma(x)$  is the same as that of  $x$ . In particular,  $x$  and  $\sigma(x)$  have the same initial path-component  $K_0 := K_0(x) = K_0(\sigma(x))$

and the same terminal path-component  $K_m := K_m(x) = K_m(\sigma(x))$ , where  $m := m(x) = m(\sigma(x))$ .

We distinguish the same mutually exclusive cases as above.

- Case 1:  $K_0 \neq K$ . Then  $\tau(x) = \sigma^\ell(x)$  and  $\tau(\sigma(x)) = \sigma^\ell(\sigma(x)) = \sigma^{\ell+1}(x)$ . We thus have  $(\sigma \circ \tau)(x) = \sigma(\sigma^\ell(x)) = \sigma^{\ell+1}(x) = \tau(\sigma(x)) = (\tau \circ \sigma)(x)$ .
- Case 2:  $K_m = K$ , that is,  $x \in C$ . Then  $\sigma(x) \in C$  and therefore  $\tau(\sigma(x)) = \sigma(x) = \sigma(\tau(x))$ .
- Case 3:  $K_0 = K \neq K_m$ . Denoting by  $b = b(x) \in \mathbb{Z}$  the unique coordinate such that  $x_b \in K$  and  $x_{b+1} \notin K$ , we have  $b(\sigma(x)) = b(x) - 1$ . Moreover,  $T(\sigma(x)) = (t, (i'_j)_{j=1}^t)$ , where  $i'_j = i_j - 1$  for all  $j = 1, \dots, t$ . Consequently for all  $i \in \mathbb{Z}$ , the value of  $\tau(\sigma(x))_i$  is obtained from formula (6.1) just by replacing  $i_j$  by  $i'_j$  for all  $j = 1, \dots, t$ . But this is exactly the value of  $\sigma(\tau(x))_i$ . We deduce that  $\tau(\sigma(x)) = \sigma(\tau(x))$ .

This shows that  $(\tau \circ \sigma)(x) = (\sigma \circ \tau)(x)$  for all  $x \in X$ . Therefore, we have  $\tau \circ \sigma = \sigma \circ \tau$ .

**Step 3.** Let  $x \in X$  and  $y := \tau(x)$ . We claim that  $y_0$  only depends on the word  $x_{[-N, \ell]} \in V^{N+\ell+1}$ . We distinguish the following three cases as above.

- Case 1:  $x_0 \in V \setminus K$ . Then  $y_0 = \tau(x)_0 = \sigma^\ell(x)_0 = x_\ell$  only depends on  $x_0$  and  $x_\ell$ .
- Case 2:  $x_\ell \in K$ . Then  $y_0 = x_0$  only depends on  $x_\ell$  and  $x_0$ .
- Case 3:  $x_0 \in K$  and  $x_\ell \in V \setminus K$ . Let  $0 \leq b < \ell$  such that  $x_b \in K$  and  $x_{b+1} \in V \setminus K$ . Recalling that the pair  $T(x) = T(x_{b-N}x_{b-N+1} \cdots x_b)$  only depends on  $x_{[b-N, b]}$ , the value  $y_0$  equals one of  $x_0, x_{dn}, x_{2dn}, \dots, x_\ell$ . In particular,  $y_0$  only depends on  $x_{[b-N, \ell]}$ .

This completes the proof that  $\tau(x)_0$  only depends on the word  $x_{[-N, \ell]}$ .

**Step 4.** It follows from the previous step that there is a map  $\mu: V^{N+\ell+1} \rightarrow V$  such that  $\tau(x)_0 = \mu(x_{[-N, \ell]})$  for all  $x \in X$ . As  $\tau$  commutes with the shift (Step 2), we deduce from Proposition 2.5 that  $\tau: X \rightarrow X$  is an endomorphism of  $X$ .

**Step 5.** Let us show that the map  $\tau: X \rightarrow X$  is surjective. Let  $x \in X$ . We distinguish the following three cases as above.

- Case 1:  $K_0 \neq K$ . Then  $\tau(x) = \sigma^\ell(x)$  so that  $x = \sigma^{-\ell}(\tau(x)) = \tau(\sigma^{-\ell}(x))$ .
- Case 2:  $K_m = K$ , that is,  $x \in C$ . Then  $x = \tau(x)$ .
- Case 3:  $K_0 = K \neq K_m$ . As usual, let  $b \in \mathbb{Z}$  such that  $x_b \in K$  and  $x_{b+1} \notin K$ . Consider the configuration  $z$  defined by setting, for all  $i \in \mathbb{Z}$ ,

$$z_i := \begin{cases} x_i & \text{if } i \leq b \\ v_j & \text{if } i = b + j \bmod n, \ b \leq i \leq b + \ell \\ x_{i-\ell} & \text{if } i \geq b + \ell, \end{cases}$$

where  $\pi = v_0v_1 \cdots v_n$  is a cycle of length  $n$  based at  $v_0 = x_b$  (cf. Lemma 6.1). It is clear that  $z \in X$ ,  $T(z) = (1, b)$  (cf. end of Remark 6.4) and  $\tau(z) = x$ .

This shows that  $\tau$  is surjective.

## 7. PROOFS OF THE MAIN RESULTS

In this section, we give the proofs of the theorems stated in the introduction.

**7.1. An auxiliary result.** We start by establishing a result that will be used in the proof of each of the theorems stated in the introduction.

Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. We denote by  $\mathcal{C}$  the poset of irreducible components of  $X$ . Given  $C, C' \in \mathcal{C}$ , we denote by  $X_{C,C'} \subset X$  the subset consisting of all configurations  $x \in X$  such that  $C_\alpha(x) = C$  and  $C_\omega(x) = C'$ . Observe that  $X_{C,C'}$  is a shift-invariant (but in general not closed) subset of  $X$  and that one has  $X_{C,C'} \neq \emptyset$  if and only if  $C \preceq C'$ . We denote by  $\mathcal{O}_{C,C'} := \{\mathcal{O}(x) : x \in X_{C,C'}\}$  the set of shift-orbits in  $X_{C,C'}$ .

**Lemma 7.1.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Let  $C$  and  $C'$  be finite irreducible components of  $X$ . Suppose that  $C$  and  $C'$  are adjacent with  $C \prec C'$ . Then the set  $\mathcal{O}_{C,C'}$  is finite.*

*Proof.* We may assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . Let  $K$  (resp.  $K'$ ) denote the path-component of  $G$  associated with  $C$  (resp.  $C'$ ). Thus, for  $x \in X_{C,C'}$  the chain of path-components is  $\Gamma(x) = \{K, K'\}$  and it follows from (3.1) that there exist integers  $b_0 < a_1$  such that  $x_i \in K$  for  $i \leq b_0$ ,  $x_i \in T$  for  $b_0 < i < a_1$ , and  $x_i \in K'$  for  $a_1 \leq i$ . Consider the word  $w(x) := ut_1 \cdots t_k v \in V^{k+2}$ , where  $u := x_{b_0} \in K$ ,  $t_j := x_{b_0+j} \in T$  for  $j = 1, 2, \dots, k := a_1 - b_0 - 1$ , and  $v := x_{a_1} \in K'$ . Recall (cf. Proposition 3.1.(i)) that the vertices  $t_1, \dots, t_k$  are all distinct. Moreover, we clearly have  $w(\sigma(x)) = w(x)$  for all  $x \in X_{C,C'}$ . As the orbit of  $x \in X_{C,C'}$  is entirely determined by  $w(x)$  (cf. Corollary 3.8), it follows that  $|\mathcal{O}_{C,C'}| = |\{w(x) : x \in X_{C,C'}\}|$ . Since the sets  $K, K'$ , and  $T$  are finite, we deduce that the set  $\mathcal{O}_{C,C'}$  is itself finite.  $\square$

**7.2. Proof of Theorem 1.1.** We shall prove the equivalence of the four conditions stated in Theorem 1.1 by successively showing (a)  $\iff$  (c), (c)  $\iff$  (b), and (c)  $\iff$  (d).

*Proof of (a)  $\iff$  (c) in Theorem 1.1.* We can assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ .

Suppose that  $X$  is surjunctive. We first show that every irreducible component of  $X$  is extremal. Suppose by contradiction that there exists  $C \in \mathcal{C}$  which is not extremal. Then there exist irreducible components  $C', C'' \in \mathcal{C}$  such that  $C' \prec C \prec C''$ . Since the poset  $\mathcal{C}$  is finite, we can take  $C'$  and  $C''$  adjacent to  $C$ . Let  $K_0$  (resp.  $K_1$ , resp.  $K_2$ ) denote the path-component of  $G$  associated with  $C'$  (resp.  $C$ , resp.  $C''$ ). Observe that  $K_0 \prec K_1 \prec K_2$  with  $K_0$  and  $K_2$  both adjacent to  $K_1$ . Denote by  $Y$  the nonempty set consisting of all configurations  $x \in X$  whose chain of path components is  $\Gamma(x) = \{K_0, K_1, K_2\}$ . Consider the endomorphism  $\tau := \tau_C : X \rightarrow X$  defined in Section 5. As  $\tau$  is injective and the subshift  $X$  is surjunctive,  $\tau$  is surjective. It follows from Remark 5.1 that  $\tau^{-1}(Y) \subset Y$ . As  $\tau$  is surjective, we deduce that  $\tau(Y) = Y$ . For every  $x \in Y$ , denote by  $b_0(x) < a_1(x) \leq b_1(x) < a_2(x)$  the integers satisfying (3.1) and set  $m(x) := b_1(x) - a_1(x)$ . It follows from (5.1) (here  $h = 0$ ) that  $a_1(\tau(x)) = a_1(x) - \ell$  and  $b_1(\tau(x)) = b_1(x)$ . Thus, we have

$$(7.1) \quad m(\tau(x)) = b_1(\tau(x)) - a_1(\tau(x)) = b_1(x) - (a_1(x) - \ell) = m(x) + \ell,$$

for all  $x \in Y$ . For a subset  $Z \subset Y$  set  $m(Z) := \min_{x \in Z} m(x)$ . Then, keeping in mind that  $Y \neq \emptyset$ , from (7.1) we get  $m(\tau(Y)) = m(Y) + \ell > m(Y)$ . This yields  $\tau(Y) \subsetneq Y$ , a contradiction. We deduce that all irreducible components of  $X$  are extremal.

Let us now show that every non-isolated irreducible component of  $X$  is finite. Let  $C \in \mathcal{C}$  be a non-isolated irreducible component. We know that  $C$  is extremal. Suppose that  $C$  is maximal (the proof in the case when  $C$  is minimal is similar). Let then  $C_0 \in \mathcal{C}$  be an irreducible component adjacent to  $C$ , so that  $C_0 \prec C$ . Set  $K_0 := K_{C_0} \in \mathcal{K}$  and  $K_1 := K_C \in \mathcal{K}$ , and observe that  $K_0 \prec K_1$  with  $K_0$  and  $K_1$  adjacent. Set now  $Y := X_{C_0, C}$ . Thus,  $Y$  is the nonempty set consisting of all configurations  $x \in X$  whose chain of path components is  $\Gamma(x) = \{K_0, K_1\}$ . Consider the endomorphism  $\tau := \tau_C: X \rightarrow X$  defined in Section 5. As  $\tau$  is injective and the subshift  $X$  is surjunctive,  $\tau$  is surjective. It follows from Remark 5.1 that  $\tau^{-1}(Y) \subset Y$ . As  $\tau$  is surjective, we deduce that  $\tau(Y) = Y$ . Therefore  $\tau^k(Y) = Y$  for all  $k \in \mathbb{N}$ .

Let  $y \in Y$ . Denote by  $b_0 < a_1$  (here  $b_0 = b_0(y)$  and  $a_1 = a_1(y)$ ) the integers satisfying (3.1) with respect to the configuration  $y$  and set  $v := y_{a_1} \in K_1$ . Let then  $\pi_v = c'_0 \cdots c'_\ell$  be the preferred cycle based at  $v$  of length  $\ell$  as in Step 1. Recall that it is entirely contained in  $K_1$ . For every  $k \in \mathbb{N}$  there exists  $x = x(k) \in Y$  such that  $\tau^k(x) = y$ . By applying (5.1) we get  $y_i = \tau^k(x)_i = c'_j$  for all  $a_1 \leq i \leq a_1 + k\ell$  with  $i \equiv a_1 + j \pmod{\ell}$ . As  $k \in \mathbb{N}$  was arbitrary, we deduce that in fact

$$(7.2) \quad y_i = c'_j \text{ for all } i \geq a_1 \text{ with } i \equiv a_1 + j \pmod{\ell}.$$

Let  $z' = z'(y) \in C$  denote the periodic configuration defined by setting  $z'_i := c'_j$  for all  $i \in \mathbb{Z}$  with  $i \equiv a_1 + j \pmod{\ell}$ . Keeping in mind (7.2), we have that  $\omega(y) = \omega(z')$ . Thus from Proposition 2.1 we deduce that

$$(7.3) \quad |\omega(y)| = |\omega(z')| = |\mathcal{O}(z')| = \ell \text{ for all } y \in Y.$$

Suppose  $C$  is infinite. Then by Proposition 3.9,  $C$  contains infinitely many periodic points and we can find  $z \in \text{Per}(C)$  whose minimal period  $p$  satisfies  $p > \ell$ . Consider the configuration  $y^z \in Y$  defined as follows. Since  $K_1$  is a path-component, there exist an integer  $k \geq 0$  and a path  $v_0 \cdots v_k$  within  $K_1$  such that  $v_0 = x_{a_1}$  and  $v_k = z_0$ . We then set, for every  $i \in \mathbb{Z}$ ,

$$(7.4) \quad y_i^z := \begin{cases} x_i & \text{if } i < a_1 \\ v_{i-a_1} & \text{if } a_1 \leq i \leq a_1 + k \\ z_{i-k-a_1} & \text{if } a_1 + k \leq i. \end{cases}$$

It is clear that  $\Gamma(y^z) = \{K_0, K_1\}$ , so that  $y^z \in Y$ . As above, the  $\omega$ -limit set  $\omega(y^z)$  of  $y^z$  equals  $\omega(z) = \mathcal{O}(z)$ . This yields  $|\omega(y^z)| = |\omega(z)| = |\mathcal{O}(z)| = p > \ell$ , contradicting (7.3). We deduce that  $C$  is finite. Thus, that every non-isolated component in  $X$  is finite. This completes the proof of (a)  $\implies$  (c).

Conversely, suppose that  $X$  satisfies (c). Let us show that  $X$  is surjunctive. Let  $\tau: X \rightarrow X$  be an injective endomorphism. Note that, for every integer  $m \geq 1$ , the map  $\tau^m$  is an injective endomorphism as well and that  $\tau$  is surjective if  $\tau^m$  is surjective. On the other hand, it follows from Proposition 3.16 that the map  $C \mapsto \tau(C)$ ,  $C \in \mathcal{C}$ , is a permutation

of  $\mathcal{C}$ . As a consequence, up to possibly replacing  $\tau$  by  $\tau^m$  for  $m := |\mathcal{C}|!$ , we may assume that  $\tau(C) = C$  for every  $C \in \mathcal{C}$ .

Let  $y \in X$ . Suppose first that  $y \in \text{NW}(X)$ , that is,  $y \in C$  for some  $C \in \mathcal{C}$ . As  $\tau(C) = C$ , there exists  $x \in C$  such that  $\tau(x) = y$ .

Suppose now that  $y \in X \setminus \text{NW}(X)$ . Then the irreducible components  $C := C_\alpha(y)$  and  $C' := C_\omega(y)$  satisfy  $C \prec C'$  by Proposition 3.4. Thus  $C$  and  $C'$  are non-isolated. Therefore,  $C$  and  $C'$  are adjacent and both finite by our assumptions. Consider the nonempty shift-invariant subset  $X_{C,C'} \subset X$  consisting of all  $x \in X$  such that  $C_\alpha(x) = C$  and  $C_\omega(x) = C'$ . Observe that  $\tau$  induces, by restriction, an injective map from  $X_{C,C'}$  into itself (cf. Proposition 3.14.(ii)). As  $\tau$  commutes with the shift, it induces a map  $\eta: \mathcal{O}_{C,C'} \rightarrow \mathcal{O}_{C,C'}$ . We claim that  $\eta$  is injective. Indeed, if  $z, z' \in X_{C,C'}$  have respective shift-orbits  $O = \mathcal{O}(z)$  and  $O' = \mathcal{O}(z')$  satisfying  $\eta(O) = \eta(O')$ , then there exists  $n \in \mathbb{Z}$  such that  $\tau(z) = \sigma^n(\tau(z'))$ . This implies  $\tau(z) = \tau(\sigma^n(z'))$  and hence  $z = \sigma^n(z')$ , since  $\tau$  is injective. We deduce that  $O = O'$ . This proves our claim. As  $\mathcal{O}_{C,C'}$  is finite by Lemma 7.1,  $\eta$  is surjective. Thus, there exists  $x \in X_{C,C'}$  such that  $\eta(\mathcal{O}(x)) = \mathcal{O}(y)$ . We deduce that there exists  $n \in \mathbb{N}$  such that  $y = \tau(\sigma^n(x))$ .

Thus  $y \in \tau(X)$ , showing that  $\tau$  is surjective. We deduce that  $X$  is surjunctive. This shows (c)  $\implies$  (a).  $\square$

*Proof of (c)  $\implies$  (b) in Theorem 1.1.* Suppose (c). Let  $\tau: X \rightarrow X$  be a surjective endomorphism and let us show that  $\tau$  is pre-injective. Note that for every integer  $m \geq 1$ , the map  $\tau^m$  is a surjective endomorphism as well and that  $\tau$  is pre-injective if  $\tau^m$  is pre-injective (cf. [9, Corollary 2.6]). It follows from Proposition 3.15 that the map  $C \mapsto \tau(C)$ ,  $C \in \mathcal{C}$ , is a permutation of  $\mathcal{C}$ . As a consequence, up to possibly replacing  $\tau$  by  $\tau^m$  for  $m := |\mathcal{C}|!$ , we may assume that  $\tau(C) = C$  for every  $C \in \mathcal{C}$ .

Let  $x, y \in X$  be homoclinic configurations such that  $\tau(x) = \tau(y)$ . By using Proposition 2.2 and Proposition 3.2, we deduce that  $C_\alpha(x) = C_\alpha(y)$  and  $C_\omega(x) = C_\omega(y)$ . Let us set  $C := C_\alpha(x) = C_\alpha(y)$  and  $C' := C_\omega(x) = C_\omega(y)$ .

If  $C = C'$ , then  $x, y \in C$  (cf. Proposition 3.2). As  $C$  is an irreducible subshift of finite type and  $\tau$  induces, by restriction, a surjective endomorphism of  $C$ , the Garden of Eden theorem for irreducible subshifts of finite type (cf. [11, Corollary 2.19] and [8, Exercise 6.120]) ensures that  $x = y$ .

Otherwise, the irreducible components  $C$  and  $C'$  satisfy  $C \prec C'$  by Proposition 3.4. Thus  $C$  and  $C'$  are non-isolated, and therefore adjacent and both finite by (c). Consider the shift-invariant subset  $X_{C,C'} \subset X$  consisting of all  $z \in X$  such that  $C_\alpha(z) = C$  and  $C_\omega(z) = C'$ . Observe that  $X_{C,C'} \neq \emptyset$  since  $x, y \in X_{C,C'}$ . It follows from Proposition 3.14.(ii) that  $\tau^{-1}(X_{C,C'}) = X_{C,C'}$ . As  $\tau$  is surjective, we deduce that  $\tau(X_{C,C'}) = X_{C,C'}$ . Consider the set  $\mathcal{O}_{C,C'}$  of shift-orbits in  $X_{C,C'}$ . Since  $\tau$  commutes with the shift, it induces a surjective map  $\eta: \mathcal{O}_{C,C'} \rightarrow \mathcal{O}_{C,C'}$ . As  $\mathcal{O}_{C,C'}$  is finite by Lemma 7.1,  $\eta$  is injective. Since  $\tau(x) = \tau(y)$ , we have  $\eta(\mathcal{O}(x)) = \mathcal{O}(\tau(x)) = \mathcal{O}(\tau(y)) = \eta(\mathcal{O}(y))$  so that  $\mathcal{O}(x) = \mathcal{O}(y)$ . We deduce that there exists  $n \in \mathbb{Z}$  such that  $x = \sigma^n(y)$ . This implies  $\tau(y) = \tau(x) = \tau(\sigma^n(y)) = \sigma^n(\tau(y))$  and hence  $n = 0$ , since the configurations in  $X_{C,C'}$  are wandering (cf. Proposition 3.4.(vii)) and therefore non-periodic. Consequently,  $x = y$ . This shows that  $\tau$  is pre-injective.  $\square$

**Lemma 7.2.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Suppose that one of the following conditions is satisfied:*

- (C1)  *$X$  admits a minimal irreducible component  $C$  which is finite and two irreducible components  $C'$  and  $C''$  such that  $C \prec C' \prec C''$ ;*
- (C2)  *$X$  admits a maximal irreducible component  $C$  which is finite and two irreducible components  $C'$  and  $C''$  such that  $C'' \prec C' \prec C$ .*

*Then  $X$  is not Moore.*

*Proof.* We can assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . We give the proof of the lemma assuming (C1) (the proof when (C2) is satisfied is similar).

Let  $K$  (resp.  $C'$ , resp.  $C''$ ) denote the path-component of  $G$  associated with  $C$  (resp.  $C'$ , resp.  $C''$ ). By our hypotheses, there exists a configuration  $y \in X$  in which appears a path in  $G$  of the form

$$v_0 t_1 \cdots t_q v'_0 \cdots v'_r t'_1 \cdots t'_s v''_0$$

with  $v_0 \in K$ ,  $v'_0, \dots, v'_r \in K'$ ,  $v''_0 \in K''$ ,  $t_1, \dots, t_q, t'_1, \dots, t'_s \in V \setminus (K \cup K' \cup K'')$ , and  $q, r, s \geq 0$ . After suppressing cycles in  $K'$  that possibly appear in  $y$ , we can assume that the vertices  $v'_0, \dots, v'_r$  are all distinct.

Choose an integer  $n \geq 1$  such that there exist a cycle  $v_0 v_1 \cdots v_{n-1} v_0$  of length  $n$  in  $K$  and a cycle  $v'_0 u'_1 \cdots u'_{n-1} v'_0$  of length  $n$  in  $K'$  (cf. Step 1 in Section 5).

Consider the paths  $\pi_0$  and  $\pi_1$  in  $G$  of length  $\ell := n + q + 1$  going from  $v_0$  to  $v'_0$  respectively defined by

$$\pi_0 := v_0 t_1 \cdots t_q v'_0 u'_1 \cdots u'_{n-1} v'_0 \text{ and } \pi_1 := v_0 v_1 \cdots v_{n-1} v_0 t_1 \cdots t_q v'_0.$$

Define a map  $\tau: X \rightarrow X$  as follows. Let  $x \in X$ . If the word  $\pi_0$  appears in  $x$ , i.e., there exists  $i \in \mathbb{Z}$  such that  $x_{[i, i+\ell]} = \pi_0$  (note that such an  $i$  is then unique), we define  $\tau(x)$  as being the configuration in  $X$  obtained from  $x$  by replacing  $\pi_0$  by  $\pi_1$  on  $[i, i + \ell]$ . Otherwise, we set  $\tau(x) := x$ .

The map  $\tau: X \rightarrow X$  is an endomorphism of  $X$  since it clearly satisfies Condition (b) in Proposition 2.5 for  $m := \ell$ .

We claim that  $\tau$  is surjective but not pre-injective.

To prove surjectivity, let  $z \in X$ . If the word  $\pi_0$  does not appear in  $z$ , then  $z = \tau(z)$ . Otherwise, consider the unique  $i \in \mathbb{Z}$  such that  $z_{[i, i+\ell]} = \pi_0$ . Since  $C$  is minimal and finite, we then have  $z_{[i-n, i+\ell-n]} = \pi_1$  by Corollary 3.8. Consider the configuration  $z' \in X$  obtained from  $z$  by replacing  $\pi_1$  by  $\pi_0$  on  $[i-n, i + \ell - n]$ . We then have  $z = \tau(z')$ . This shows that  $\tau$  is surjective.

To prove that  $\tau$  is not pre-injective, consider the configuration  $y' \in X$  obtained from  $y$  by replacing the path  $\pi_1$  on  $[i-n, i + \ell - n]$  by the path  $\pi_0$ . Then  $\tau(y') = y = \tau(y)$ . As  $y$  and  $y'$  are distinct homoclinic configurations, we deduce that  $\tau$  is not pre-injective.

Since  $\tau$  is a surjective endomorphism of  $X$  which is not pre-injective, we conclude that  $X$  is not Moore.  $\square$

**Lemma 7.3.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Suppose that  $X$  contains two irreducible components  $C$  and  $C'$  with  $C \prec C'$  such that one of these components is infinite and the other is finite and extremal. Then  $X$  is not Moore.*

*Proof.* We can assume that  $X$  is the vertex subshift associated with a finite directed graph  $G = (V, E)$ . We give the proof assuming that  $C$  is infinite and  $C'$  is finite and maximal (the proof when  $C'$  is infinite and  $C$  is finite and minimal is similar).

Let  $K$  (resp.  $K'$ ) denote the path-component of  $G$  associated with  $C$  (resp.  $C'$ ). Since  $C \prec C'$ , there exists a configuration  $\bar{x} \in X$  whose initial and terminal path-components are  $K$  and  $K'$ , respectively. Then, up to shifting  $\bar{x}$  if necessary, there exist  $v_0 \in K$ ,  $u_0 \in K'$ ,  $t_1, \dots, t_q \in V \setminus (K \cup K')$ , where  $q \geq 0$ , such that  $\bar{x}_{[0, q+1]} = v_0 t_1 \cdots t_q u_0$ .

As  $C$  is infinite, it follows from Lemma 6.1 that there exists an integer  $n \geq 1$  and two distinct cycles  $v_0 v_1 \cdots v_{n-1} v_0$  and  $v_0 v'_1 \cdots v'_{n-1} v_0$  in  $K$  based at  $v_0$  of length  $n$ . In fact, arguing as in Step 1 in Section 5, up to replacing  $n$  by a suitably chosen multiple, we can also find a cycle  $u_0 u_1 \cdots u_{n-1} u_0$  in  $K'$  based at  $u_0$  of the same length  $n$ .

Consider the paths  $\pi_0$ ,  $\pi'_0$ , and  $\pi_1$  in  $G$ , with length  $\ell := n + q + 1$  and going from  $v_0$  to  $u_0$ , respectively defined by

$$\begin{aligned}\pi_0 &:= v_0 v_1 \cdots v_{n-1} v_0 t_1 \cdots t_q u_0 \\ \pi'_0 &:= v_0 v'_1 \cdots v'_{n-1} v_0 t_1 \cdots t_q u_0 \\ \pi_1 &:= v_0 t_1 \cdots t_q u_0 u_1 \cdots u_{n-1} u_0.\end{aligned}$$

Define a map  $\tau: X \rightarrow X$  as follows. Let  $x \in X$ . If there exists  $i \in \mathbb{Z}$  such that  $x_{[i, i+\ell]}$  equals either  $\pi_0$  or  $\pi'_0$  (note that such an  $i$  is then unique), we define  $\tau(x)$  as being the configuration in  $X$  obtained from  $x$  by replacing  $\pi_0$  or  $\pi'_0$  by  $\pi_1$  on  $[i, i + \ell]$ . If there is no such  $i$ , we set  $\tau(x) := x$ .

The map  $\tau: X \rightarrow X$  is an endomorphism of  $X$  since it clearly satisfies Condition (b) in Proposition 2.5 with  $m := \ell$ .

We claim that  $\tau$  is surjective but not pre-injective. Let  $z \in X$ . If neither  $\pi_0$  nor  $\pi'_0$  appears in  $z$ , then  $z = \tau(z)$  is in the image of  $\tau$ . Otherwise, consider the unique  $i \in \mathbb{Z}$  such that  $z_{[i, i+\ell]} = \pi_0$  (resp.  $z_{[i, i+\ell]} = \pi'_0$ ). Since  $C'$  is finite and extremal, we have  $z_{[i+n, i+\ell+n]} = \pi_1$  by Corollary 3.8. Consider now the configuration  $w \in X$  obtained from  $z$  by replacing  $\pi_1$  by  $\pi_0$  on  $[i + n, i + \ell + n]$ . We then have  $z = \tau(w)$ . This shows that  $\tau$  is surjective.

In order to show that  $\tau$  is not pre-injective, we observe that the configuration  $\bar{x} \in X$  satisfies that  $\bar{x}_{[0, \ell]} = \pi_1$ . This follows from the fact that  $C'$  is finite and maximal, by using again Corollary 3.8. Consider the configurations  $y, y' \in X$  obtained from  $\bar{x}$  by replacing  $\pi_1$  by  $\pi_0$  and  $\pi'_0$  on  $[0, \ell]$ , respectively. We have  $y \neq y'$  since  $\pi_0 \neq \pi'_0$ . Since  $y$  and  $y'$  are homoclinic and satisfy  $\tau(y) = \bar{x} = \tau(y')$ , we deduce that  $\tau$  is not pre-injective.

Since  $\tau$  is an endomorphism of  $X$  which is surjective but not pre-injective, we conclude that  $X$  is not Moore.  $\square$

**Lemma 7.4.** *Let  $A$  be a finite set and let  $X \subset A^{\mathbb{Z}}$  be a subshift of finite type. Suppose that  $X$  contains an infinite irreducible component which is extremal but not isolated. Then  $X$  is not Moore.*

*Proof.* Let  $C$  denote an infinite irreducible component of  $X$  which is extremal but not isolated. Suppose that  $C$  is minimal (the case when  $C$  is maximal is similar). Since  $C$  is not isolated, we can find an irreducible component  $C'$  of  $X$  such that  $C \prec C'$ . Consider the endomorphism  $\tau = \tau^C$  defined in Section 6. In Step 5 in that section, when proving surjectivity of  $\tau$ , given a configuration  $x \in X_{C,C'}$  we defined a configuration  $z \in X$  such that  $\tau(z) = x$ . Keeping the same notation therein, we define another configuration  $z'$  by setting, for all  $i \in \mathbb{Z}$ ,

$$z'_i := \begin{cases} x_i & \text{if } i \leq b \\ v'_j & \text{if } i = b + j \bmod n, \ b \leq i \leq b + \ell \\ x_{i-\ell} & \text{if } i \geq b + \ell, \end{cases}$$

where now  $\pi' = v'_0 v'_1 \cdots v'_n$  is a cycle of length  $n$  based at  $v'_0 = x_b$  and distinct from  $\pi$  (recall that  $C$  is infinite so that Lemma 6.1 applies). It is clear that  $z' \in X$ , it is distinct from  $z$  (since  $\pi' \neq \pi$ ), it is homoclinic to  $z$  ( $z$  and  $z'$  agree on  $\mathbb{Z} \setminus [b, b + \ell]$ ),  $T(z') = (1, b) = T(z)$  and  $\tau(z') = x = \tau(z)$ . We deduce that  $\tau$  is not pre-injective.

Since  $\tau$  is an endomorphism of  $X$  which is surjective but not pre-injective, we conclude that  $X$  is not Moore.  $\square$

*Proof of (b)  $\implies$  (c) in Theorem 1.1.* Suppose that  $X$  contains infinite irreducible components which are not isolated. If there exists one of these that is extremal, we deduce from Lemma 7.4 that  $X$  is not Moore. Otherwise, we can find two irreducible components  $C, C'$  with  $C$  infinite and  $C'$  finite and minimal, such that  $C' \prec C$ . It then follows from Lemma 7.3 that  $X$  is not Moore either. Suppose now that all infinite irreducible components of  $X$  are isolated and that there is an irreducible component which is not extremal. It then follows from Lemma 7.2 that  $X$  is not Moore. This shows (b)  $\implies$  (c).  $\square$

*Proof of (c)  $\iff$  (d) in Theorem 1.1.* Let  $P$  denote the perfect kernel of  $X$ . Let  $\mathcal{C}$  denote the poset of irreducible components of  $X$ . Denote by  $\mathcal{C}_\infty$  (resp.  $\mathcal{C}_{\infty,i}$ , resp.  $\mathcal{C}_f$ , resp.  $\mathcal{C}_{f,i}$ ) the set of irreducible components of  $X$  that are infinite (resp. infinite and isolated, resp. finite, resp. finite and isolated).

Suppose (c). Thus,  $\mathcal{C}_\infty = \mathcal{C}_{\infty,i}$  and  $X \setminus (\bigcup_{C \in \mathcal{C}_\infty} C) = \bigcup_{C_1, C_2 \in \mathcal{C}_f} X_{C_1, C_2}$ . We then deduce from Proposition 3.11 that the derived set of  $X$  is the non-wandering subshift of finite type  $X' = \bigcup_{C \in \mathcal{C}_\infty \cup (\mathcal{C}_f \setminus \mathcal{C}_{f,i})} C$ . It follows that  $X^{(n)} = \bigcup_{C \in \mathcal{C}_\infty} C$  for every  $n \geq 2$ . Therefore, we have  $P = \bigcup_{C \in \mathcal{C}_\infty} C$  and  $\text{rk}_{\text{CB}}(X) \leq 2$ . This shows that (c) implies (d).

Conversely, suppose (d). Let  $C$  be an infinite irreducible component of  $X$ . It follows from Proposition 3.11 that  $C' = C$ . Consequently, we have  $C = C' \subset X'$  and, recursively,  $C \subset X^{(n)}$  for all  $n \in \mathbb{N}$ . We deduce that  $C \subset P$ . Since,  $P = \bigcup_{C \in \mathcal{C}_{\infty,i}} C$ , we deduce that  $C$  is isolated.

It follows from Proposition 3.10 that the subset  $Y := X \setminus P$  is closed. We thus have  $X^{(n)} = P \sqcup Y^{(n)}$  for all  $n \in \mathbb{N}$ . In particular,  $P \sqcup Y^{(\text{rk}_{\text{CB}}(X))} = P^{(\text{rk}_{\text{CB}}(X))} \sqcup Y^{(\text{rk}_{\text{CB}}(X))} = X^{(\text{rk}_{\text{CB}}(X))} = P$ , which yields  $Y^{(\text{rk}_{\text{CB}}(X))} = \emptyset$ . Since  $\text{rk}_{\text{CB}}(X) \leq 2$ , we thus have  $Y'' = \emptyset$ . As  $Y$  is countable and compact, this implies that  $Y'$  is finite. As  $Y'$  is  $\sigma$ -invariant, every configuration  $y$  in  $Y'$  is periodic, so that  $C_\alpha(y) = C_\omega(y)$ . On the other hand, it follows

from Proposition 3.11 that the set  $Y \setminus Y'$  of isolated configurations in  $Y$  consists of all  $y \in Y$  such that  $C_\alpha(y)$  and  $C_\omega(y)$  are both finite, with  $C_\alpha(y)$  minimal and  $C_\omega(y)$  maximal. Therefore  $Y$  is the disjoint union of the set of finite extremal components of  $X$ , and of the subset consisting of all  $y \in X$  such that  $C_\alpha(y) \prec C_\omega(y)$  with  $C_\alpha(y)$  and  $C_\omega(y)$  both finite,  $C_\alpha(x)$  minimal, and  $C_\omega(x)$  maximal. Since isolated (finite or infinite) components are extremal, we deduce that every irreducible component of  $X$  is extremal. This shows that (d) implies (c).  $\square$

*Proof of (d)  $\iff$  (e) in Theorem 1.1.* We keep the notation from the proof of (c)  $\iff$  (d).

Suppose (d). It follows from Proposition 3.3.(iii) that the perfect kernel  $P = \bigcup_{C \in \mathcal{C}_\infty} C$  of  $X$  is non-wandering. Moreover, being a finite disjoint union of subshifts of finite type (Proposition 3.4.(iv)),  $P$  is itself a subshift of finite type. Set  $Y := X \setminus P$ . It follows from the Cantor-Bendixson decomposition theorem that  $Y$  is countable. As it is closed (by Proposition 3.10) and shift-invariant,  $Y$  is a subshift. In fact, keeping in mind that all infinite irreducible components are isolated, we have that  $Y$  is the vertex subshift associated with  $V \setminus \bigcup_{\substack{K \in \mathcal{K} \\ C_K \in \mathcal{C}_\infty}} K$ , so it is a subshift of finite type as well. As  $\text{rk}_{\text{CB}}(X) \leq 2$ , we have  $P = X'' = (X')' = ((P \sqcup Y)')' = (P \sqcup Y')' = P \sqcup Y''$ . We deduce that  $Y'' = \emptyset$ , so that  $\text{rk}_{\text{CB}}(Y) \leq 2$ . This shows that (d) implies (e).

Conversely, assume (e). Then  $X = Z \sqcup Y$ , where  $Z \subset \text{NW}(X)$  and  $Y := X \setminus Z$  are subshifts of finite type with  $Y$  countable and  $\text{rk}_{\text{CB}}(Y) \leq 2$ , respectively. Since  $Y$  is countable with  $\text{rk}_{\text{CB}}(Y) \leq 2$ , we have  $Y'' = \emptyset$ . On the other hand, for  $C \in \mathcal{C}$ , we have  $C' = C$  if  $C \in \mathcal{C}_\infty$  and  $C' = \emptyset$ , otherwise. Consequently,  $Z' = \bigcup_{\substack{C \in \mathcal{C} \\ C \subset Z}} C' = \bigcup_{\substack{C \in \mathcal{C} \\ C \subset Z}} C$  and therefore  $Z'' = Z'$ . We deduce that  $X'' = Z'' \cup Y'' = Z''$ , so that  $\text{rk}_{\text{CB}}(X) \leq 2$ , and  $P = X'' = Z'' = Z'$ . We claim that  $P = \bigcup_{C \in \mathcal{C}_{\infty,i}} C$ . Indeed, suppose that there exists  $C \in \mathcal{C}_\infty$  which is not isolated. Then there exists  $C' \in \mathcal{C}$  such that either  $C' \prec C$  or  $C \prec C'$ . Accordingly, either  $X_{C',C}$  or  $X_{C,C'}$  is contained in  $X \setminus \text{NW}(X) \subset Y$ . Since  $C$  is infinite, the corresponding set is uncountable, contradicting the countability of  $Y$ . Thus every irreducible component  $C \in \mathcal{C}_\infty$  is isolated. Moreover, every irreducible component  $C \in \mathcal{C}_\infty$  is uncountable and since  $Y$  is countable, one has  $C \in Z$ . We deduce that  $P = \bigcup_{\substack{C \in \mathcal{C}_\infty \\ C \subset Z}} C = \bigcup_{\mathcal{C}_{\infty,i}} C$  and the claim follows.  $\square$

**7.3. Proof of Theorem 1.3.** We shall prove the equivalence of the three conditions stated in Theorem 1.3 by successively showing (a)  $\iff$  (b) and (b)  $\iff$  (c).

*Proof of (a)  $\iff$  (b) in Theorem 1.3.* Suppose that  $X$  is injunctive. By Proposition 4.5, every irreducible component of  $X$  is finite. As  $X$  is Moore since it is injunctive, we then deduce from Lemma 7.2 that every irreducible component of  $X$  is extremal. This shows (a)  $\implies$  (b).

Conversely, suppose that  $X$  satisfies (b), i.e., every irreducible component of  $X$  is finite and extremal. Let  $\tau: X \rightarrow X$  be a surjective endomorphism of  $X$ . We want to show that  $\tau$  is injective. Let  $\mathcal{C}$  denote the set of irreducible components of  $X$ . Up to replacing  $\tau$  by  $\tau^m$  for  $m := |\mathcal{C}|!$ , it follows from Proposition 3.15 that we may assume that  $\tau(C) = C$

for every  $C \in \mathcal{C}$ . Let  $x, y \in X$  such that  $\tau(x) = \tau(y)$ . If  $x \in C$  for some  $C \in \mathcal{C}$ , then  $\tau(y) = \tau(x) \in C$  and hence  $y \in C$ . As  $C$  is finite and  $\tau$  is surjective, the restriction of  $\tau$  to  $C$  is injective, so that  $x = y$ . Otherwise, setting  $C := C_\alpha(x), C' := C_\omega(x) \in \mathcal{C}$ , we have  $C \prec C'$  and  $x \in X_{C,C'}$ . Observe that  $\tau$  induces, by restriction, a surjective map from  $X_{C,C'}$  onto itself (cf. Proposition 3.14.(ii)). As  $\tau$  commutes with the shift, it induces a map  $\eta: \mathcal{O}_{C,C'} \rightarrow \mathcal{O}_{C,C'}$ . We claim that the map  $\eta$  is also surjective. Indeed, if  $O \in \mathcal{O}_{C,C'}$  then  $O = \mathcal{O}(z)$  for some  $z \in X_{C,C'}$ . If  $z' \in X_{C,C'}$  is such that  $z = \tau(z')$ , then  $O = \eta(O')$  for  $O' := \mathcal{O}(z')$ . This proves our claim. As  $\mathcal{O}_{C,C'}$  is finite by Lemma 7.1, the map  $\eta$  must be injective. As  $\tau(y) = \tau(x) \in X_{C,C'}$ , we have  $y \in X_{C,C'}$  and  $\eta(\mathcal{O}(y)) = \eta(\mathcal{O}(x))$ . Thus, we have  $\mathcal{O}(y) = \mathcal{O}(x)$ . We deduce that there exists  $n \in \mathbb{Z}$  such that  $y = \sigma^n(x)$ . This implies that  $\tau(x) = \tau(y) = \tau(\sigma^n(x)) = \sigma^n(\tau(x))$ . As  $\tau(x) \in X_{C,C'}$  is wandering and therefore not periodic, we get  $n = 0$ , so that  $y = x$ . Therefore  $\tau$  is injective. This shows that  $X$  is injunctive and completes the proof of (a)  $\iff$  (b).  $\square$

*Proof of (b)  $\iff$  (c) in Theorem 1.3.* Suppose (b). Denoting by  $\mathcal{C}'$  the set of irreducible components of  $X$  that are not isolated, we then deduce from Proposition 3.11 that  $X$  has derivatives  $X' = \bigcup_{C \in \mathcal{C}'} C$  and  $X'' = \emptyset$ . Therefore,  $X$  is countable and  $\text{rk}_{\text{CB}}(X) \leq 2$ . This shows that (b) implies (c).

Conversely, suppose (c). As  $X$  is countable, we have  $X^{(\text{rk}_{\text{CB}}(X))} = \emptyset$ . Since  $\text{rk}_{\text{CB}}(X) \leq 2$ , we deduce that  $X'' = \emptyset$ . By compactness, this implies that  $X'$  is finite. As  $X'$  is  $\sigma$ -invariant, every configuration in  $X'$  is periodic. On the other hand, it follows from Proposition 3.11 that the set  $X \setminus X'$  of isolated configurations in  $X$  consists of all  $x \in X$  such that  $C_\alpha(x)$  and  $C_\omega(x)$  are both finite, with  $C_\alpha(x)$  minimal and  $C_\omega(x)$  maximal. Therefore,  $X \setminus X'$  is the disjoint union of the set of finite extremal components of  $X$  and of the subset  $Y \subset X$  consisting of all  $x \in X$  such that  $C_\alpha(x) \prec C_\omega(x)$  with  $C_\alpha(x)$  and  $C_\omega(x)$  both finite,  $C_\alpha(x)$  minimal, and  $C_\omega(x)$  maximal. We deduce that every irreducible component of  $X$  is finite and extremal. This shows that (c) implies (b).  $\square$

## 8. EXAMPLES

It follows from our results that if  $A$  is a finite set and  $X \subset A^{\mathbb{Z}}$  is a subshift of finite type then  $X$  is of one of the following types:

- Type 1:  $X$  is surjunctive, injunctive, Moore, and Myhill;
- Type 2:  $X$  is surjunctive, injunctive, and Moore, but not Myhill;
- Type 3:  $X$  is surjunctive, Moore, and Myhill, but not injunctive;
- Type 4:  $X$  is surjunctive and Moore, but neither injunctive nor Myhill;
- Type 5:  $X$  is neither surjunctive, nor injunctive, nor Moore, nor Myhill.

The examples of subshifts of finite type below show that each of these five types can occur.

**Example 8.1** (Type 1). Let  $A := \{0\}$  and let  $X := A^{\mathbb{Z}} = \{0^{\mathbb{Z}}\}$  denote the full shift over  $A$ . Observe that  $X$  is a subshift of finite type admitting  $\emptyset$  as a defining set of forbidden words. The identity map  $\text{Id}_X$  is the unique endomorphism of  $X$ . As  $\text{Id}_X$  is bijective,  $X$  is of Type 1. Note that  $X$  is irreducible. The perfect kernel of  $X$  is the empty set and we have  $\text{rk}_{\text{CB}}(X) = 1$ .

**Example 8.2** (Type 1). Let  $A := \{0, 1\}$  and consider the subshift  $X := \{0^{\mathbb{Z}}, 1^{\mathbb{Z}}\} \subset A^{\mathbb{Z}}$  consisting of the two periodic configurations of minimal period 2. Observe that  $X$  is a subshift of finite type admitting  $\{00, 11\}$  as a defining set of forbidden words. On the other hand, the endomorphisms of  $X$  are its two permutations. As every endomorphism of  $X$  is bijective,  $X$  is surjunctive, injunctive, Moore, and Myhill. Note that  $X$  is irreducible. The perfect kernel of  $X$  is the empty set and we have  $\text{rk}_{\text{CB}}(X) = 1$ .

**Example 8.3** (Type 2). Let  $A := \{0, 1\}$  and consider the subshift  $X := \{0^{\mathbb{Z}}, 1^{\mathbb{Z}}\} \subset A^{\mathbb{Z}}$  consisting of the two constant configurations. Observe that  $X$  is a subshift of finite type admitting  $\{01, 10\}$  as a defining set of forbidden words. As  $X$  is finite, it is surjunctive, injunctive, and it has the Moore property. On the other hand, the endomorphism  $\tau: X \rightarrow X$  defined by  $\tau(0^{\mathbb{Z}}) = \tau(1^{\mathbb{Z}}) = 0^{\mathbb{Z}}$  is pre-injective but not surjective. Consequently,  $X$  does not have the Myhill property. Note that  $X$  is non-wandering. Its irreducible components are  $\{0^{\mathbb{Z}}\}$  and  $\{1^{\mathbb{Z}}\}$ , and they are both isolated. The perfect kernel of  $X$  is the empty set and we have  $\text{rk}_{\text{CB}}(X) = 1$ .

**Remark 8.4.** The subshifts described in the two previous examples have homeomorphic underlying topological spaces. As one of these subshifts is Myhill while the other is not, this shows that the Myhill property is not a topological invariant for subshifts of finite type.

**Example 8.5** (Type 3). Let  $A := \{0, 1\}$  and let  $X := A^{\mathbb{Z}}$  denote the full shift over  $A$ . Observe that  $X$  is a subshift of finite type admitting  $\emptyset$  as a defining set of forbidden words. By the original Moore-Myhill Garden of Eden theorem [20], [21],  $X$  is both Moore and Myhill. As  $X$  is Myhill, it is surjunctive. However,  $X$  is not injunctive. This follows from Theorem 1.3 since  $X$  is infinite and irreducible. Actually, an explicit surjective but not injective endomorphism of  $X$  is provided by the map  $\tau: X \rightarrow X$  given by  $\tau(x)_i := x_i + x_{i+1} \pmod{2}$  for all  $x \in X$  and  $i \in \mathbb{Z}$ . Consequently,  $X$  is of Type 3. Note that  $X$  is irreducible. The perfect kernel of  $X$  is  $X$  itself and we have  $\text{rk}_{\text{CB}}(X) = 0$ .

**Example 8.6** (Type 4). Let  $A := \{0, 1, 2\}$  and consider the subshift  $X := \{0, 1\}^{\mathbb{Z}} \cup \{2^{\mathbb{Z}}\}$ . Observe that  $X$  is a subshift of finite type admitting  $\{02, 20, 12, 21\}$  as a defining set of forbidden words. It has two irreducible components, namely  $\{0, 1\}^{\mathbb{Z}}$  and  $\{2^{\mathbb{Z}}\}$ . These two irreducible components are isolated. We thus deduce from Theorem 1.1 that  $X$  is surjunctive and has the Moore property. On the other hand, it follows from Theorem 1.3 that  $X$  is not injunctive. Actually, an explicit surjective but not injective endomorphism of  $X$  is provided by the map  $\tau: X \rightarrow X$  given by  $\tau(x)_i := x_i + x_{i+1} \pmod{2}$  for all  $x \in \{0, 1\}^{\mathbb{Z}}$  and  $i \in \mathbb{Z}$ , and  $\tau(2^{\mathbb{Z}}) = 2^{\mathbb{Z}}$ . Moreover,  $X$  is not Myhill since the map  $\tau: X \rightarrow X$  given by  $\tau(x) := x$  for all  $x \in \{0, 1\}^{\mathbb{Z}}$  and  $\tau(2^{\mathbb{Z}}) := 0^{\mathbb{Z}}$  is a pre-injective endomorphism of  $X$  which is not surjective. Consequently,  $X$  is of Type 4. The perfect kernel of  $X$  is  $\{0, 1\}^{\mathbb{Z}}$  and we have  $\text{rk}_{\text{CB}}(X) = 1$ .

**Example 8.7** (Type 5). Let  $A := \{0, 1, 2\}$  and consider the subshift of finite type  $X \subset A^{\mathbb{Z}}$  admitting  $\{10, 20, 21\}$  as a defining set of forbidden words. Observe that the word 12 can appear at most once in a given configuration of  $X$ . The map  $\tau: X \rightarrow X$  which replaces the word 12 by the word 11 is an injective endomorphism that is not surjective.

Therefore,  $X$  is not surjunctive. In particular, it does not have the Myhill property. On the other hand, the map  $\tau': X \rightarrow X$  which replaces the word 112 by the word 122 is a surjective endomorphism which is not pre-injective (the configurations  $\dots 00.1222\dots$  and  $\dots 00.1122\dots$  are homoclinic and have the same image under  $\tau'$ ). Thus,  $X$  does not have the Moore property. It follows that  $X$  is not injunctive either. Consequently,  $X$  is of Type 5. Note that  $X$  fails to be non-wandering. It has three irreducible components, namely,  $\{0^{\mathbb{Z}}\}$ ,  $\{1^{\mathbb{Z}}\}$ , and  $\{2^{\mathbb{Z}}\}$ , and they satisfy  $\{0^{\mathbb{Z}}\} \prec \{1^{\mathbb{Z}}\} \prec \{2^{\mathbb{Z}}\}$ . The perfect kernel of  $X$  is the empty set and we have  $\text{rk}_{\text{CB}}(X) = 3$ .

Here is another example of a subshift of finite type of Type 5. Contrary to the previous example, this one is uncountable.

**Example 8.8** (Type 5). Let  $A := \{0, 1, 2\}$  and consider the subshift of finite type  $X \subset A^{\mathbb{Z}}$  admitting  $\{02, 20, 21\}$  as a defining set of forbidden words. Observe that the word 12 can appear at most once in a given configuration  $x \in X$ . The map  $\tau: X \rightarrow X$  which replaces the word 12 by the word 11 is an injective endomorphism that is not surjective (the word 012 appears in  $X$  but not in  $\tau(X)$ ). Therefore,  $X$  is not surjunctive. In particular, it does not have the Myhill property. On the other hand, the map  $\tau': X \rightarrow X$  which replaces the word 112 by the word 122 is a surjective endomorphism which is not pre-injective (the configurations  $\dots 00.1222\dots$  and  $\dots 00.1122\dots$  are homoclinic and have the same image under  $\tau'$ ). Thus,  $X$  does not have the Moore property. It follows that  $X$  is not injunctive either. Consequently,  $X$  is of Type 5. Note that  $X$  fails to be non-wandering. It has two irreducible components, namely,  $\{0, 1\}^{\mathbb{Z}}$  and  $\{2^{\mathbb{Z}}\}$ , and they satisfy  $\{0, 1\}^{\mathbb{Z}} \prec \{2^{\mathbb{Z}}\}$ . The perfect kernel of  $X$  is  $X$  itself and we have  $\text{rk}_{\text{CB}}(X) = 0$ .

**Remark 8.9.** The subshift described in the previous example and the full shift  $\{0, 1\}^{\mathbb{Z}}$  of Example 8.5 have underlying topological spaces that are homeomorphic (they are both non-empty perfect compact metrizable spaces). We deduce that surjunctivity (resp. Moore) is not a topological invariant for subshifts of finite type.

We describe now an example showing that injunctivity is not a topological invariant for sofic subshifts.

**Example 8.10.** Let  $A := \{0, 1, 2, 3, 4, 5\}$ . For each  $k \in \{0, 1, 2\}$ , let  $x^{(k)} \in A^{\mathbb{Z}}$  denote the configuration defined by  $x_0^{(k)} := 2k + 1$  and  $x_i^{(k)} := 2k$  for all  $i \in \mathbb{Z} \setminus \{0\}$ . Consider the subshift  $Y_k \subset A^{\mathbb{Z}}$  given by  $Y_k := \{(2k)^{\mathbb{Z}}\} \cup \mathcal{O}(x^{(k)})$ . Each  $Y_k$  is topologically conjugate to the sunny-side-up subshift and hence is sofic [8, Exercise 1.90]. Consequently, the subshift  $Z := Y_0 \sqcup Y_1 \sqcup Y_2$  is also sofic since it is a finite union of sofic subshifts [8, Exercise 1.101]. One checks that  $Z$  is injunctive (a surjective endomorphism of  $Z$  is also injective since it must permute the three constant configurations and the orbits of the  $x^{(k)}$ s). On the other hand, we have  $Z' = \{0^{\mathbb{Z}}, 2^{\mathbb{Z}}, 4^{\mathbb{Z}}\}$ , so that the underlying topological space of  $Z$  is homeomorphic to the underlying topological space of the subshift of finite type  $X$  of Example 8.7 since they have the same Mazurkiewicz-Sierpinski characteristic  $(1, 3)$ . As  $Z$  is surjunctive while  $X$  is not, this shows that injunctivity is not a topological invariant for sofic subshifts.

## 9. OPEN QUESTIONS

It is shown in [9] that non-wandering Smale systems are surjunctive. The characterization of surjunctivity remains open for general Smale systems.

**Question 1.** Is every surjunctive Smale system topologically conjugate to the disjoint union of a non-wandering Smale system and a countable subshift of finite type of Cantor-Bendixson rank at most 2?

We note that a positive answer to this question would imply the well-known conjecture that every Anosov diffeomorphism on a compact manifold is non-wandering (see, e.g., [5]). Indeed, every self-embedding of a compact manifold is surjective (by Brouwer’s invariance of domain), and a non-empty open subset of a manifold cannot be countable.

Similarly, one may ask for the characterization of the Moore, Myhill and injunctivity properties of general Smale systems. It is reasonable to conjecture that general Smale systems satisfy similar properties as subshifts of finite type, that the Myhill property still implies the Moore property, and that the Moore property is equivalent to surjunctivity.

For subshifts of finite type, the Myhill property remains without a clean characterization.

**Problem.** Give a characterization of the Myhill property for  $\mathbb{Z}$ -subshifts of finite type.

As shown by Fiorenzi, the even subshift is Myhill (and therefore surjunctive) but not Moore [11], which already shows that sofic subshifts behave quite differently from subshifts of finite type.

**Question 2.** Which sofic subshifts are surjunctive (resp. injunctive, resp. Moore, resp. Myhill)?

It is possible to construct compact dynamical  $\mathbb{Z}$ -systems that are injunctive but not surjunctive. For subshifts, we do not know any such examples.

**Question 3.** Does there exist a subshift with finite alphabet over  $\mathbb{Z}$  which is injunctive but not surjunctive?

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