Trees in positive entropy subshifts

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Abstract

We give a simple proof for the fact that positive entropy subshifts contain infinite binary trees where branching happens synchronously in each branch, and the branching times form a set with positive lower asymptotic density.

The proof is trivial! Just view it as a
rational metric space
whose elements are
countable combinatorial games

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1 Introduction

In topological dynamics, topological entropy of a dynamical system measures the information in orbits, by counting the exponential growth rate of different partial orbits up to some accuracy. In a binary subshift, one way entropy can arise is that, somewhere in partial orbits, we literally see all 2^n distinct binary patterns up to some n. One may ask to what extent entropy always arises from such "free choices". In particular, the notion of independence entropy [6] of a subshift measures how much entropy comes from individual Cartesian subhypercubes contained in A^n as $n \to \infty$, and positive entropy indeed implies positive independence entropy. Here, we prove a strengthening of the result in the context of one-dimensional subshifts.

Theorem 1. *A subshift $X \subset \Sigma^{\mathbb{N}}$ has positive entropy if and only if it contains a steadily branching binary tree.*

We can interpret $\Sigma^{\mathbb{N}}$ as a $|\Sigma|$-ary tree and X a closed subset of it(s boundary). It is in this sense that X contains the binary tree. Steadily branching means that the embedded binary tree branches at a uniform sequence of times which form a set of positive lower asymptotic density.

We note that countable subshifts can have growth rate arbitrarily close to exponential, and obviously only contain trees with finitely many branchings. It is also easy to find examples where it is not possible to find a tree where branchings happen at a syndetic set of times.

The proof is straightforward using standard results about the winning shift, which is the set of winning turn orders in a certain word-building game associated to X. The winning shift is known to have the same entropy as X. This implies it has a point with positive density, and this point is the branching structure of a steadily branching tree in X.

It is easy to deduce a two-sided version as well. For $x \in \Sigma^{-\mathbb{N}}$ and $y \in \Sigma^{\mathbb{N}}$ write $x \cdot y \in \Sigma^{\mathbb{Z}}$ for the configuration with x and y back-to-back.

Theorem 2. *A subshift $X \subset \Sigma^{\mathbb{Z}}$ has positive entropy if and only if for some $x \in \Sigma^{-\mathbb{N}}$, the set $\{y \in \Sigma^{\mathbb{N}} \mid x \cdot y \in X\}$ contains a steadily branching binary tree.*

The winning shift shift was introduced in [8], and has also been studied in [7]. We found out while working on [7] that a concept equivalent to the winning shift was discovered in the set systems setting already in 2002 [1] (even if ostensibly only for binary words), and we found out while working on the present paper that [1] has been applied by dynamicists [5, 2], and their proof that this gives positive independence entropy boils down to the same proof we give here.

Nevertheless, we feel Theorem 1 is an interesting statement about subshifts, the statement does not to our knowledge appear in the literature, and its proof through winning shifts is worth making explicit.

Question 1. *Can the notion of winning shift be extended to general expansive systems (or beyond)?*

2 The winning shift

Let $\mathbb{N} \ni 0$ be the natural numbers. For $z, b \in \mathbb{N}^{\mathbb{N}}$, denote $z \leq b \iff \forall i : z_i \leq b_i$.

Definition 1. *Let Σ be a finite alphabet and let $X \subset \Sigma^{\mathbb{N}}$. A* tree in X with branching structure *$b \in \mathbb{N}^{\mathbb{N}}$ is a set of sequences $(x^z)_{\mathbb{N}^{\mathbb{N}} \ni z \leq b}$, where each x^z is an element of X, and*

$$x^z_{[0,i)} = x^{z'}_{[0,i)} \iff z_{[0,i)} = z'_{[0,i)}.$$

A steadily branching binary tree *is a tree with branching structure $b \in \{0,1\}^{\mathbb{N}}$ satisfying* $\liminf_{i \to \infty} \frac{\sum_{j=0}^{i-1} b_j}{i} > 0$.

In words, for distinct sequences $z, z' \in \mathbb{N}^{\mathbb{N}}$, the first position where x^z and $x^{z'}$ differ is the same as the first position where z and z' differ. For binary b, this means the nonzero positions in b are the positions where our tree must branch in two, and more generally $b_i = n$ means the tree must branch n times in position i, explaining why we call this sequence a branching structure.

Definition 2. *Let $X \subset \Sigma^{\mathbb{N}}$ be a subshift. Let $W(X)$, the* winning shift *of X, be the set of all branching structures of trees in X.*

This is defined in a game-theoretic framework in [8]: a tree in X with branching structure z can be interpreted as a winning strategy for the first player in a word-building game where on the ith turn, the first player picks a set of $z_i + 1$ symbols, and the second picks on of them, and the first player wins if the word obtained in the limit belongs in X.

It is shown in [8] that the winning shift is indeed a subshift. A subshift $Y \subset \mathbb{N}^{\mathbb{N}}$ is *down* if for all $y' \in \mathbb{N}^{\mathbb{N}}$ we have

$$(y \in Y \wedge \forall i : y'_i \leq y_i) \implies y' \in Y.$$

The *words of a subshift* $X \subset A^{\mathbb{N}}$ are the words $w \in A^*$ such that $x|_{[0,|w|-1]} = w$ for some $x \in X$. The words of a subshift form its *language*. The following fact is Proposition 5.7 of [8] (the down part is trivial). The notation differs slightly, and in the reference $\tilde{W}(X)$ is used for what we call $W(X)$.

Lemma 1 ([8])**.** *The subshift $W(X)$ has the same number of words of each length as X, and is down.*

For a finite word $w \in \mathbb{N}^*$, write $\sum w = \sum_i w_i$ for the sum of the symbols in w, and $|w|$ for the length of w as a word. The key to finding steadily branching trees is to study the *density* $\sum w/|w|$ of a word w. If $Y \subset \{0,1\}^{\mathbb{N}}$ is a subshift, by Y^k we mean the Cartesian power with the diagonal action, with alphabet $\{0,1\}^k$.

Lemma 2. *If the subshift X has positive entropy and is down, then for some $\beta > 0$ there exist arbitrarily long words w of $X \cap \{0,1\}^{\mathbb{N}}$ with $\sum w/|w| \geq \beta$.*

Proof. Let $\{0, 1, ..., k-1\}$ be the alphabet of X. Let

$$b(X) = \{x \in \{0,1\}^{\mathbb{N}} \mid \exists y \in X : \forall i : y_i \neq 0 \iff x_i \neq 0\}.$$

Since X is down, it is easy to see that the number of words in X of length n is at most the number of words in $b(X)^k$ of length n. Thus, if X has positive entropy, so does $b(X)^k$, and thus so does $b(X)$. Suppose thus that the number of words in $b(X)$ of every length n is at least α^n for some $\alpha > 0$, as is clearly implied by positive entropy.

The number of words of length n with at most k many 1s is at most

$$\binom{n}{k} 2^k < \left(\frac{n \cdot e}{k}\right)^k 2^k.$$

Setting $k = \beta n$ this becomes $((\frac{n \cdot e}{\beta n})^\beta 2^\beta)^n$ and we observe that as $\beta \to 0$ we have $2^\beta \to 1$ and $(\frac{n \cdot e}{\beta n})^\beta = (\frac{e}{\beta})^\beta \to 1$, so for small enough $\beta > 0$, we have $(\frac{n \cdot e}{\beta n})^\beta 2^\beta < \alpha$, and thus there must be words of length n in $b(X)$ with at least βn many 1s, for arbitrarily large n. These are words of $X \cap \{0,1\}^{\mathbb{N}}$ since X is down. □

3 The proofs

Proof of Theorem 1. Obviously a steadily branching tree implies positive entropy.

For the other direction, let $\Sigma \subset \mathbb{N}$ be a finite alphabet and $Y \subset \Sigma^{\mathbb{N}}$ a subshift. Write s_n for the maximal sum $\sum w$ of a word w of length n in Y. This sequence is clearly subadditive, so $\lim_n s_n/n$ exists, say $\lim_n s_n/n = \alpha$.

We outline the usual addendum[1] that there must be a configuration $y \in Y$ such that every prefix w of y satisfies $\sum w/|w| \geq \alpha$.

[1] The author learned this from [4, 3] and P. Guillon. It is not clear what the best reference is.

Suppose not. Then every point X has a prefix w such that $\sum w/|w| < \alpha$, and this must happen after a bounded number of steps by compactness, thus there exists $\epsilon > 0$ such that for some m we always find a prefix w of length at most m in any point $y \in Y$, such that $\sum w/|w| < \alpha - \epsilon$. Now, given any long word w of Y we can split it as $w = w_0 w_1 \cdots w_k$ with $|w_k| \leq m$ and $\sum w_i/|w_i| < \alpha - \epsilon$ for all i. If w is long enough, then since $\sum uv/|uv| \leq \max(\sum u/|u|, \sum v/|v|)$ for all words u, v, we have

$$\sum w/|w| < \alpha - \epsilon/2$$

for all long enough words, a contradiction to $\lim s_n/n = \alpha$.

Now set $Y = W(X)$ and apply Lemma 2. The lemma implies that $\alpha > 0$ in the above. The point $y \in W(X)$ whose density stays above α gives the braching structure of a steadily branching tree in X. □

Proof of Theorem 2. Let $X_R \subset \Sigma^{\mathbb{N}}$ be the subshift of right tails of points in X and apply Theorem 1. Let $y \in \{0,1\}^{\mathbb{N}}$ be the branching structure of some steadily branching tree and α the lower asymptotic density. It is easy to see that for any n, we can find a finite prefix w of $y = wx$ of length at least n such that for all prefixes u of x of length at most n, $\sum u/|u| \geq \alpha$. Otherwise, by cutting long words into ones of length at most n as in the previous argument, we see that the lower asymptotic density is less than α.

The fact we can find such wx as a branching structure implies that the tree corresponding to x can follow at least one word of length $|w|$. Letting n tend to infinity, by compactness we obtain a steadily branching tree that can follow some left tail. □

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