Realizing Subshifts as Sets of Surjective and Injective Nonuniform Cellular Automata*

Ville Salo[1]

TUCS – Turku Center for Computer Science,
University of Turku, Finland,
vosalo@utu.fi

Abstract. Given a finite set of local rules, the sequences built from them give a set of nonuniform cellular automata when the ith coordinate is computed by the local rule located at i in the sequence. It is known [2] that for any such set of rules, the subset of nonuniform cellular automata obtained which give a surjective global map is in fact a sofic subshift. For the injective sequences, it is only known [2] that the set is ζ-automatic (accepted by a Büchi automaton), and it need not be open or closed. We prove a preliminary result on implementing sofic subshifts as surjective and injective sequences: we provide the required set of local rules for any SFT. In fact, we prove that SFTs are exactly the sets which are simultaneously the set of surjective and injective sequences of nonuniform cellular automata for some set of local rules.

Keywords: subshifts, nonuniform cellular automata

1 Introduction

Nonuniform cellular automata are a vast relaxation to the usual definition of cellular automata (CA): while CA are defined as the (countably many) shift-commuting continuous functions on the space $\Sigma^{\mathbb{Z}}$, 'nonuniform cellular automaton' technically just refers to an arbitrary continuous function. Since this set is rather large and hard to work with, we usually assume the local maps at each coordinate come from a finite set of local rules. Some basic mathematical properties of nonuniform CA have been studied at least in [1].

In [2], it is proved that for a finite set of local rules Γ, the set of sequences over Γ whose corresponding nonuniform CA is surjective is a sofic subshift. The set of injective sequences (nor its complement), on the other hand, need not even be a closed set. It is only known that this set is ζ-automatic. Of course, it makes sense to ask if these solutions are optimal, in the sense that there is no natural subclass of sofic subshifts (ζ-automatic sets) such that all sets of surjective (injective) sequences are in fact in this subclass. An example in [2] answers the obvious first question of whether SFTs suffice for surjective sequences in the negative: the (proper sofic) even shift can be implemented this way.

* Research supported by the Academy of Finland Grant 131558

In this work-in-progress paper, we show that, in addition to the even shift, all SFTs can be implemented as the set of surjective sequences. For the set of local rules we give, the same SFT will be the set of injective sequences. This leads to the question of which *subshifts* can be realized as injective sequences. We answer this question completely: any closed set implementable as injective sequences is in fact an SFT.

What makes these implementation problems hard is that the local rules of the sequence cannot 'see' each other: they behave the exact same way no matter which rules are used in their neighboring cells. Thus, surjectivity or injectivity must in some way truly arise from their interplay, and as the even shift example and our results show, such interplay is indeed possible. For these reasons, we believe answering these questions might give some insight into the theory of nonuniform CA, and perhaps even the theory of cellular automata, as even surjective cellular automata are still not completely understood.

In what follows, we will state some specific questions once they become natural to ask. However, as we do not know an example of a sofic shift which is not in $\mathcal{S}_n$ for all $n > 1$, if $\mathcal{S}_n$ is in fact the set of all sofic shifts for all $n > 1$ (which we conjecture is not the case though), most of these questions would have rather trivial answers.

Since we are studying the symbolic dynamical aspects of nonuniform CA obtained from a finite set of local rules, it is worth mentioning that it is not a priori clear that the subshifts realizable through the surjectivity property are closed under conjugacy (and in fact we conjecture the contrary).

2 Definitions

For a finite set Σ, we call the space $\Sigma^{\mathbb{Z}}$ with the product topology the *full shift*, the alphabet Σ having the discrete topology. For $x \in \Sigma^{\mathbb{Z}}$, we say $w \in \Sigma^*$ *occurs in* x if $x_{[i,i+|w|-1]} = w$ for some i, and we denote this by $w \sqsubset x$. A *subshift* is a set $X \subset \Sigma^{\mathbb{Z}}$ which is closed under the left shift map $\sigma(x)_i = x_{i+1}$ and its inverse (that is, *shift-invariant*) and which is topologically closed. Subshifts are compact metrizable spaces, and we use a little bit of compactness in the proof of Theorem 2. Otherwise, we mainly use a more combinatorial description of subshifts, the existence of a (possibly infinite) set of *forbidden words* F such that X is exactly the set of sequences over Σ where no word of F occurs. If the set F in the combinatorial definition can be taken to be finite, then X is said to be *of finite type* (SFT for short). If F can be taken to be a regular language, X is said to be *sofic*.

A *local rule over* Σ is a tuple (P, Σ, f), where $P \subset \mathbb{Z}$ is finite, Σ is a finite alphabet, and f is a function from Σ^P to Σ. When Γ is a set of local rules over a single alphabet Σ, a sequence $c \in \Gamma^{\mathbb{Z}}$ is called a *nonuniform CA*. It's *global function*, also denoted c, is the function from $\Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ defined by

$$c(x)_i = f_i(x_{i+P_i}),$$

where $c_i = (P_i, \Sigma, f_i)$. Nonuniform CA are exactly the continuous functions on $\Sigma^{\mathbb{Z}}$. In this paper, only the case where Γ is finite is considered. A nonuniform

CA with a single local rule is called a *cellular automaton*. Just like nonuniform CA characterize the continuous functions, cellular automata are exactly the continuous maps that commute with σ.

A *symbol projection* is a function $\pi : \Delta \to \Gamma$ between two finite alphabets Δ and Γ. Such functions naturally extend to sequences $\delta \in \Delta^{\mathbb{Z}}$ by $\pi(\delta)_i = \pi(\delta_i)$, and we use the same notation π for the extended function. We often write $\pi(\delta)$ as $\pi\delta$ when this is clearer. A bijective symbol projection is a simple example of a *conjugacy* between two subshifts: a homeomorphism from a subshift onto another. Similarly, every symbol projection is an example of a *factor map*, a continuous shift-commuting function from one subshift onto another. Both conjugacies and factor maps are always realized by a block code (that is, a cellular automaton whose domain is not necessarily equal to its codomain) [3].

We define $\mathcal{S}_n$ as the class of subshifts $X \subset \Delta^{\mathbb{Z}}$ such that for a suitable symbol projection $\pi : \Delta \to \Gamma$ to a finite set of local rules Γ over Σ, and for $\Sigma = \{0, \ldots, n-1\}$, the global function of (the nonuniform CA) $\pi\delta$ is surjective on $\Sigma^{\mathbb{Z}}$ if and only if $\delta \in X$. We define $\mathcal{I}_n$ analogously, by requiring that $\pi\delta$ be injective. Of course, the exact n symbols in Σ do not matter. We define $\mathcal{S} = \bigcup_n \mathcal{S}_n$ and $\mathcal{I} = \bigcup_n \mathcal{I}_n$. In [2], it is proved that $X \in \mathcal{S}$ is always sofic, and that $X \in \mathcal{I}$ is ζ-automatic. We will not need the latter fact, and thus omit the definition of a ζ-automatic set. Let us, however, mention that this refers to a set accepted by a two-way Büchi automaton.

When discussing nonuniform CA, we use subshifts in three different roles. To perhaps prevent some confusion, let us clarify the situation and choose some naming conventions.

- The full shift $\Sigma^{\mathbb{Z}}$ is the subshift on which our nonuniform CA act, and we usually call Σ the set of *states*. Sequences in $\Sigma^{\mathbb{Z}}$ are called *input points*, and we use variables in the x, y, z family for them.
- We have a subshift $X \subset \Delta^{\mathbb{Z}}$ (or $Y \subset \Theta^{\mathbb{Z}}$) which we are trying to implement (or assume to be implemented), through a projection to a set of local rules, as the set of surjective or injective nonuniform CA. We use the variable δ and its variants for points of $\Delta^{\mathbb{Z}}$.
- Implicitly, we use the subshift of nonuniform CA obtained through a projection π from Δ to a set of local rules Γ. We only use sequences of such a subshift through π by $\pi\delta$ where $\delta \in \Delta^{\mathbb{Z}}$.

3 Nonuniform CA

We start by mentioning some closure properties and relations between the classes $\mathcal{S}_n$: first of all, trivially, $\mathcal{S}$ is closed under symbol permutations (by changing the projection π). The following lemma follows almost as easily.

Lemma 1. *The class $\mathcal{S}$ is closed under intersection.*

Proof. Let $X \subset \Delta^{\mathbb{Z}}$ be in $\mathcal{S}_m$ with the projection π, and $Y \subset \Delta^{\mathbb{Z}}$ in $\mathcal{S}_n$ with the projection π'. We let $\pi'' : \Delta \to \Gamma$ be the map $d \mapsto (\pi d, \pi' d)$, where $(\pi d, \pi d')$ is the local map over $\Sigma = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ that behaves as πd on the $\{0, \ldots, m-1\}$ component and as $\pi' d$ on the $\{0, \ldots, n-1\}$ component. It is then easy to see that $\pi'' \delta$ is surjective for $\delta \in \Delta^{\mathbb{Z}}$ if and only if both $\pi\delta$ and $\pi'\delta$ are surjective, and π'' thus realizes the subshift $X \cap Y$. $\square$

Question 1. Is $\mathcal{S}_n$ closed under intersection for a fixed $n > 1$?

Later (in Lemma 5) we will see that $\mathcal{S}$ is at least closed under symbol disjoint union. However, the general case seems to be harder:

Question 2. Is $\mathcal{S}$ closed under union?

There is at least the following simple connection between the classes $(\mathcal{S}_i)_{i \in \mathbb{N}}$, which says (assuming we restrict the class of subshifts in $\mathcal{S}$ to a set) that $n \mapsto \mathcal{S}_n$ is an order preserving function from $(\mathbb{N}, |)$ to $((\mathcal{S}_i)_{i \in \mathbb{N}}, \subset)$.

Lemma 2. *If $m|n$, then $\mathcal{S}_m \subset \mathcal{S}_n$.*

Proof. By splitting the set of states Σ into two tracks, one of size m and one of size $\frac{n}{m}$, we can have all the local maps preserve the $\frac{n}{m}$ track and operate as the appropriate local map on the track of size m. This way, all subshifts in $\mathcal{S}_m$ can be realized in $\mathcal{S}_n$. $\square$

Again, this might be a vacuous observation, since we cannot answer the following:

Question 3. Are the sets $\mathcal{S}_n$ equal for all $n > 1$?

We note a somewhat disconnected, but rather curious closure property:

Definition 1. *Let $X \subset \Delta^{\mathbb{Z}}$ be a subshift, let $p \in \mathbb{Z}$, and let*

$$Y = \{\delta \in \Delta^{\mathbb{Z}} \mid (\delta_{i+kp})_{k \in \mathbb{Z}} \in X \ \forall i\}.$$

Then Y is called the p-interleaving of X.

Lemma 3. *The classes $\mathcal{S}_n$ are closed under p-interleaving for all p.*

Proof. Let $X \subset \Delta^{\mathbb{Z}}$ be in $\mathcal{S}_n$ by the symbol projection $\pi : \Delta \to \Gamma$. If we change each local rule $(P, \Sigma, f) \in \Gamma$ to $(p \cdot P, \Sigma, f)$ where $p \cdot P = \{p \cdot j \mid j \in P\}$, then all the local rules work as previously, but on p separate tracks (separated by the residue classes of cell indices modulo p). $\square$

Now, let us proceed to our main results: SFTs are in $\mathcal{S} \cap \mathcal{I}$ and they are the only subshifts in $\mathcal{I}$.

Definition 2. *Let $\mathcal{F}$ be any field. We write I_n for the identity matrix with size $n \times n$, and J_n for the matrix*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

also of size $n \times n$, where $a = (-1)^{n-1}$. This is the sum of the identity matrix with its rotated version, with the bottom left value possibly negated depending on the parity of n.

Definition 3. *Let (M_1, M_2) be a pair of matrices, such that $M_1 = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $M_2 = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. For $w \in \{0,1\}^n$, we write $[M_1, M_2]_w$ for the matrix $\begin{pmatrix} [u_1, v_1]_{w_1} \\ \vdots \\ [u_n, v_n]_{w_n} \end{pmatrix}$, where we write $[a,b]_0 = a$ and $[a,b]_1 = b$.*

Lemma 4. $\det[J_n, I_n]_w = 0$ *if and only if $w = 0^n$ (with any underlying field).*

Proof. If $w \neq 0^n$, then when computing the determinant (straight from the permutation definition), only the identity permutation appears with nonzero coefficient, and $\det[J_n, I_n]_w = 1$. On the other hand, clearly $\det[J_n, I_n]_{0^n} = \det J_n = 0$, since only two permutations with nonzero coefficients appear and their signs cancel each other. □

Theorem 1. *For all m, every SFT $X \subset \Delta^{\mathbb{Z}}$ is in $\mathcal{S}_m \cap \mathcal{I}_m$, even for a single choice of local maps.*

Proof. We prove that there exists a projection $\pi : \Delta \to \Gamma$ such that for $\delta \in \Delta^{\mathbb{Z}}$, the nonuniform CA $\pi\delta$ is surjective on $\Sigma^{\mathbb{Z}}$ where $\Sigma = \{0, \ldots, m-1\}$ if and only if it is injective if and only if $\delta \in X$. First, let us assume m is a prime number, and let $\mathbb{F}_m$ be the finite field of order m. We explain the local rules πd for $d \in \Delta$ by specifying how they differ from the identity function.

Let X be defined by the finite set of forbidden words F. We may assume $|u| = n$ for all $u \in F$ by extending the shorter forbidden words in all possible ways. For each word $u \in F$, we choose a word $u' = 01^k0$ such that all the u' are distinct and $k > n$. Now, in any input the occurrences of words of the form $u'vu'$ where $|v| = n$ are disjoint in their v-parts, and a local map can easily detect if such a pattern occurs around the cell it is rewriting.

The idea is to construct the local maps in such a way that only words of the form $u'vu'$ are changed, and only in their v-part, and the image does not depend on anything outside the word. Then, a nonuniform CA is surjective (or equivalently injective) if and only if for every u', the block of n local maps mapping the v-part of $u'vu'$ gives a bijective map.

We obtain these mappings from J_n and I_n (taken over the field $\mathbb{F}_m$) as follows: If the local map πd finds itself rewriting the ith coordinate of the subword $v \in \Sigma^n$ in an occurrence $u'vu'$ in the input point $x \in \Sigma^{\mathbb{Z}}$, it checks whether $u_i = d$. If this is the case, πd outputs $(J_n v^T)_i$, and otherwise it outputs $(I_n v^T)_i$.

Now, consider the mapping among words of the form $u'vu'$ when a block of local maps $\pi d_1, \ldots, \pi d_n$ is rewriting v. Let $w \in \{0,1\}^n$ be defined by $w_i = 0$ if $d_i = u_i$, and $w_i = 1$ otherwise. Then, $u'vu'$ is rewritten to $u'([J_n, I_n]_w v^T)^T u'$, so the mapping is not bijective if and only if $w = 0^n$ by Lemma 4.

So, consider a sequence $\delta \in \Delta^{\mathbb{Z}}$. If $\delta_{[j,j+n-1]} = u$ for some $u \in F$, then δ is not surjective, by considering input points x where $x_{[j-|u'|,j+n-1+|u'|]} = u'vu'$. On the other hand, if u does not occur, δ is clearly bijective, as all rewritings happen in patterns of the form $u'vu'$ with $|v| = n$, and the mapping on the v-part is bijective by the previous paragraph.

In the case when m is not prime, we simply separate a track of prime alphabet size p as in the proof of Lemma 2 and construct the local rules as above for this track, while behaving as identity on the other track (of alphabet size $\frac{m}{p}$). $\square$

We say $X \subset \Delta^{\mathbb{Z}}$ and $Y \subset \Theta^{\mathbb{Z}}$ are symbol disjoint if $\Delta \cap \Theta = \emptyset$, and in this case their union $X \cup Y \subset (\Delta \cup \Theta)^{\mathbb{Z}}$ is called their *symbol disjoint union.*

Lemma 5. *Subshifts in $\mathcal{S}$ are closed under symbol disjoint union.*

Proof. Let $X \subset \Delta^{\mathbb{Z}}$ and $Y \subset \Theta^{\mathbb{Z}}$ be in $\mathcal{S}$ with $\Delta \cap \Theta = \emptyset$. By Lemma 2, they are in fact both in $\mathcal{S}_n$ for some n by projections $\pi : \Delta \to \Gamma$ and $\pi' : \Theta \to \Gamma'$, respectively.

Consider the symbol map $\pi'' : (\Delta \cup \Theta) \to (\Gamma \cup \Gamma')$, the piecewise map equal to π on Δ and equal to π' on Θ. Obviously, if $\delta \in (\Delta \cup \Theta)^{\mathbb{Z}}$ only contains symbols from Δ (Θ), $\pi''\delta$ is surjective if and only if the nonuniform CA $\pi\delta$ ($\pi'\delta$) is surjective. Thus, we have proved that Z is in $\mathcal{S}$ for some Z such that $Z \cap \Delta^{\mathbb{Z}} = X$ and $Z \cap \Theta^{\mathbb{Z}} = Y$. But this means also the subshift $Z \cap (\Delta^{\mathbb{Z}} \cup \Theta^{\mathbb{Z}}) = X \cup Y$ is in $\mathcal{S}$ by Theorem 1 and Lemma 1, since $(\Delta^{\mathbb{Z}} \cup \Theta^{\mathbb{Z}})$ is an SFT, and $\mathcal{S}$ is closed under intersection. $\square$

Theorem 2. *The closed sets in $\mathcal{I}$ are exactly the SFTs.*

Proof. SFTs are closed sets, and they are in $\mathcal{I}$ by Theorem 1. For the other inclusion, let a $X \subset \Delta^{\mathbb{Z}}$ be a closed set in $\mathcal{I}_n$ through the surjective projection $\pi : \Delta \to \Gamma$ where Γ is a finite set of local maps over $\Sigma = \{0, \ldots, n-1\}$. Since every set in $\mathcal{I}$ is clearly shift-invariant, X is a subshift.

We first show that there exists r such that if $\delta \in X$ and $x_0 \neq y_0$ for $x, y \in \Sigma^{\mathbb{Z}}$, then $(\pi\delta)(x)_i \neq (\pi\delta)(y)_i$ for some $i \in [-r, r]$: If this were not the case, we would find, for each r, a triple (δ^r, x^r, y^r) such that $x^r_0 \neq y^r_0$ but $(\pi\delta)(x)_{[-r,r]} = (\pi\delta)(y)_{[-r,r]}$. Now, let (δ, x, y) be any limit point of the sequence $((\delta^r, x^r, y^r))_r$ in the subshift $(\Delta \times \Sigma \times \Sigma)^{\mathbb{Z}}$. Then, we easily see that $x_0 \neq y_0$ but $(\pi\delta)(x) = (\pi\delta)(y)$, and thus $\pi\delta$ is not injective. This is a contradiction because X was assumed to be closed, πX to be exactly the set of injective nonsurjective CA over Γ, and δ to be a limit point of the sequence $(\delta^r)_r$ of injective nonsurjective CA.

Now, consider two words $uv, vw \sqsubset X$ such that $|v| \geq 2r+1$. We claim that $uvw \sqsubset X$, which then implies that X is an SFT; this is in fact a well-known characterization of SFTs [3]. Take an extension $z_\ell u.vz_r \in X$ for uv where the decimal point indicates the origin to be on its right, and an extension $z'_\ell.vwz'_r \in X$ for vw. Now, if $z_\ell u.vwz'_r$ were not injective through π, there would exist inputs $x, y \in \Sigma^{\mathbb{Z}}$ such that $x \neq y$ but $\pi(z_\ell u.vwz'_r)(x) = \pi(z_\ell u.vwz'_r)(y)$.

Let i be such that $x_i \neq y_i$. Since $|v| \geq 2r+1$, there must be at least r coordinates of v either to the left or to the right of i in $z_\ell u.vwz'_r$. By symmetry, we may assume they are to the left of i, that is $i \geq r$. But then, since $\pi(z'_\ell.vwz'_r)(x)_{[i-r,i+r]} \neq \pi(z'_\ell.vwz'_r)(y)_{[i-r,i+r]}$, we must have

$$\pi(z_\ell u.vwz'_r)(x)_{[i-r,i+r]} \neq \pi(z_\ell u.vwz'_r)(y)_{[i-r,i+r]},$$

as the local maps computing the cells $[i-r, i+r]$ are unchanged. Therefore, $uvw \sqsubset X$, and thus X is an SFT. $\square$

It is known that, contrary to the case of $\mathcal{I}$, subshifts in $\mathcal{S}$ do not coincide with SFTs, but are a proper superclass:

Proposition 1 ([2]). *The even shift with forbidden words* $10(00)^*1$ *is in* $\mathcal{S}$.

However, we do not believe this is the case for all sofic shifts, and give a simple candidate sofic shift which we do not believe is in $\mathcal{S}$.

Conjecture 1. The sofic shift with forbidden words $10^*1 + 01^*0$ is not in $\mathcal{S}$.

In particular, this would mean that $\mathcal{S}$ is not equal to the class of all sofic shifts. To emphasize our belief that outside SFTs, examples of sofic shifts in $\mathcal{S}$ are rather sporadic and dependent on the exact coordinate structure, we conjecture the following:

Conjecture 2. The class $\mathcal{S}$ is not closed under conjugacy.

Note that if $\mathcal{S}$ were closed under factor maps, all sofic shifts would be in $\mathcal{S}$ by Theorem 1 and the well-known fact the sofic shifts are exactly the closure of SFTs under factor maps [3].

We say a sofic shift $X \subset \Delta^{\mathbb{Z}}$ is in $\mathcal{S}_n$ *with linear rules* if for the obvious linear structure on $\{0, \ldots, n-1\}$, it is implemented by a projection $\pi : \Delta \to \Gamma$ to a finite set Γ of affine rules, rules that simply sum together (modulo n) multiples of the cell values seen in the pattern P. Even if all sofic shifts were in fact in $\mathcal{S}$, the following question might not have a trivial answer.

Question 4. Which sofic shifts are in $\mathcal{S}_n$ with linear rules, for a given n?

Considering affine rules would not change the answer, as adding constants modulo n to a fixed set of cells does not affect the surjectivity of a function. We ask this question because the local rules implementing the even shift [2] are linear. Note that while the local rules given by Theorem 1 are in a sense given by matrix multiplication, they are far from linear.

4 Conclusions and Future Work

We have proved some basic closure properties for the class $\mathcal{S}$, and obtained that all SFTs are in this class. We hope that these results will be of help in the exact characterization of this class of subshifts. Of course, in addition to closure properties and implementation techniques, we would need a technique for showing a sofic shift is *not* in $\mathcal{S}$ (unless all sofic shifts can in fact be implemented). In the case of $\mathcal{I}$ we were more lucky, and were able to give the exact characterization as the class of SFTs for all n.

While we have only considered the case where the nonuniform cellular automata are required to be surjective or injective in this paper, we think similar questions might have meaningful and interesting answers also in the case where the nonuniform CA are required to be surjective on a given subshift, perhaps a mixing SFT. Of course, another direction would be to consider nonuniform CA on the cells of a more complicated group than $\mathbb{Z}$, acting on a full shift on the same group. In the case of $\mathbb{Z}^d$, at least the proof of Theorem 1 should work rather directly, but it is less clear which property is actually relevant in the group.

Acknowledgements

I would like to thank Ilkka Törmä for his comments on this article, and Jyrki Lahtonen for his help with finding the final form of the matrices needed in the proof of Theorem 1.

References

1. Alberto Dennunzio, Enrico Formenti, and Julien Provillard. Non-uniform cellular automata: classes, dynamics, and decidability. *CoRR*, abs/1107.5228, 2011.
2. Alberto Dennunzio, Enrico Formenti, and Julien Provillard. Computational complexity of rule distributions of non-uniform cellular automata. In *Proceedings of the 6th international conference on Language and Automata Theory and Applications*, LATA'12, pages 204–215, Berlin, Heidelberg, 2012. Springer-Verlag.
3. Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.