Realization Problems for Nonuniform Cellular Automata*

Ville Salo

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Abstract

Given a finite set of local rules, the sequences built from them give a set of nonuniform cellular automata. It is known [Dennunzio, Formenti, Provillard, 2012] that for any such set of local rules, the subset of nonuniform cellular automata built from them which give a surjective global map is in fact a sofic subshift. For the injective sequences, this set is only ζ-automatic (accepted by a Büchi automaton), and it is not necessarily open or closed. We prove the first corresponding realization result: we provide, for any SFT, a set of local rules such that the surjective (injective) nonuniform cellular automata over them are precisely the sequences in the SFT (up to a renaming of symbols). In fact, we prove that SFTs are exactly the subshifts which are the set of injective sequences of nonuniform cellular automata for some set of local rules. We also consider surjectivity of subclasses of nonuniform cellular automata, and realizability questions for other properties, in particular number conservation and chain transitivity.

1 Introduction

Nonuniform cellular automata are a vast relaxation to the usual definition of cellular automata (CA): while CA are defined as the (countably many) shift-commuting continuous functions on the space $\Sigma^{\mathbb{Z}}$, 'nonuniform cellular automaton' technically just refers to an arbitrary continuous function. This set is rather unaccessible, and thus we usually add the constraint that local maps at each coordinate come from a finite set of local rules (a priori, this is not a dynamical notion like the others; see Proposition 14 for such a characterization). Some basic mathematical properties of nonuniform CA have been studied at least in [1].

In [2], it is proved that for a finite set of local rules Γ, the set of sequences over Γ whose corresponding nonuniform CA is surjective is a sofic subshift. The set of injective sequences (nor its complement), on the other hand, need not even be a closed set. Nevertheless, there still exists a simple automata theoretic

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description, as such a set is ζ-automatic. Of course, it makes sense to ask if these solutions are optimal, in the sense that there is no natural subclass of sofic subshifts (ζ-automatic sets) such that all sets of surjective (injective) sequences are in fact in this subclass. That is, given a sofic shift X, we consider the realization problem of whether we can choose a set of local rules so that X is (up to a renaming of symbols) the set of surjective sequences for that set of local rules. An example in [2] answers the obvious first question of whether SFTs suffice for surjective sequences in the negative: the (proper sofic) even shift can be realized as the set of surjective sequences.

In this paper, we show that, in addition to the even shift, all SFTs, and some other sofic shifts of interest, can be realized as the set of surjective sequences. We do not give examples of sofic shifts which are not realizable, except for subclasses of local rules, and only give simple examples which we cannot show to be realizable. Techniques cleverer than those developed here would be needed to prove an unrealizability result, or to prove that every sofic shift is realizable, whichever be the case. Thus, the main question we set out to solve stays open:

Question 1. *Which sofic shifts are realizable through surjectivity?*

All SFTs can also be realized as injective sequences. This leads to the question of which *subshifts* can be realized so. We answer this question completely: any closed set realizable as injective sequences is in fact an SFT.

Since we do not know any examples of sofic shifts which are not realizable as the surjective sequences, it makes sense to try to restrict our attention to subclasses of local rules. For 'asynchronous CA', that is, nonuniform CA where either the identity map or a fixed CA is used in every cell, we show nontrivial nonrealizability results, which also have some implications for the general case. We also study nonuniform CA that respect an algebraic structure, and give a general technique for showing nonrealizability results for such maps using the notion of a 'dual' of a linear nonuniform CA.

We also briefly study other dynamical properties. Interestingly, the set of (temporally) chain transitive sequences of local rules is also always a subshift, and we can realize all SFTs through this property as well. For many other dynamical properties, we give examples showing that the set of sequences with the given property is not a subshift. Some connections between nonuniform CA and symbolic dynamics are given in Section 7, where we make some general observations about nonuniform CA and discuss nonuniform CA over sofic shifts.

What makes the type of realization problems studied in this article hard (and hopefully interesting) is that the local rules of the sequence cannot 'see' each other: they behave the exact same way no matter which rules are used in their neighboring cells. Thus, surjectivity or injectivity must in some way truly arise from their interplay, and as the even shift example of [2] and our results show, quite interesting interplay is indeed possible. For this reason, we believe the study of realizability questions might give some insight into the theory of nonuniform CA, and perhaps even the theory of cellular automata, as even surjective cellular automata are still not well understood.

2 Definitions

We write $\mathbb{Z}$ for the ring of integers, and $\mathbb{N}$ for the natural numbers, including 0. The ring of integers modulo i is denoted by $\mathbb{Z}_i$.

For a finite set Σ, we call the space $\Sigma^{\mathbb{Z}}$ with the product topology the *full shift*, the finite alphabet Σ having the discrete topology. For $x \in \Sigma^{\mathbb{Z}}$ or $x \in \Sigma^*$, we say $w \in \Sigma^*$ *occurs in* x if $x_{[i,i+|w|-1]} = w$ for some i, and we denote this by $w \sqsubset x$. A *subshift* is a set $X \subset \Sigma^{\mathbb{Z}}$ which is closed under the left shift map $\sigma(x)_i = x_{i+1}$ and its inverse (that is, *shift-invariant*) and which is topologically closed. Subshifts are compact metrizable spaces. We also talk about words occurring in subshifts, and write $w \sqsubset X$ if $\exists x \in X : w \sqsubset x$. Subshifts also have the more combinatorial definition that there exists a (possibly infinite) set of *forbidden words* F such that X is exactly the set of sequences over Σ where no word of F occurs, that is, $X = \{x \in \Sigma^{\mathbb{Z}} \mid \forall w \in F : w \not\sqsubset x\}$. If the set F in the combinatorial definition can be taken to be finite, then X is said to be of finite type, or an *SFT*. If F can be taken to be a regular language, X is said to be *sofic*.

The *language* $\mathcal{B}(X)$ of a subshift X is the set of words w that occur in points of X. Often, it is more convenient to supply the 'allowed words' of a subshift instead of the forbidden ones, in the sense of supplying its language directly, using the fact that subshifts are uniquely determined by their language [3]: When L is an *extendable language*, that is, $w \in L \implies \exists u, v \in A^+ : uwv \in L$, we write $\mathrm{Fact}(L) = \{w \mid \exists u, v \in A^* : uwv \in L\}$ for the *factor closure* of L, and define

$$\mathcal{B}^{-1}(L) = \{x \in \Sigma^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, n \in \mathbb{N} : x_{[i,i+n]} \in \mathrm{Fact}(L)\}.$$

This is the smallest subshift whose language contains all the words in L. The equalities $\mathcal{B}^{-1}(\mathcal{B}(X)) = X$ and $\mathcal{B}(\mathcal{B}^{-1}(L)) = \mathrm{Fact}(L)$ hold for subshifts X and languages L. If L is defined by a regular expression, then $\mathcal{B}^{-1}(L)$ is known to be sofic.

For $a \in \Sigma$, we say $x \in \Sigma^{\mathbb{Z}}$ is a-finite if $x_i = a$ whenever $|i|$ is sufficiently large. We write ${}^\infty a^\infty$ for the point x with $x_i = a$ for all $i \in \mathbb{Z}$. Such a point x is called *unary*.

A *local rule over* Σ is a tuple (P, Σ, f), where $P \subset \mathbb{Z}$ is the finite *neighborhood*, Σ is a finite alphabet, and f is a function from $\Sigma^{|P|}$ to Σ. The minimal ℓ, r such that $P \subset [-\ell, r]$ are called the *left* and *right radius*, respectively. When $F = (P, \Sigma, f)$ is a local rule, we sometimes also write $F(a_1, \ldots, a_{|P|}) = f(a_1, \ldots, a_{|P|})$, and P and Σ are often omitted when they are clear from context. When Γ is a set of local rules over a single alphabet Σ, a sequence $c \in \Gamma^{\mathbb{Z}}$ is called a *nonuniform CA*. Its *global function*, also denoted c, is the function from $\Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ defined by

$$c(x)_i = f_i(x_{i+P_i}),$$

where $c_i = (P_i, \Sigma, f_i)$. Nonuniform CA are exactly the continuous functions on $\Sigma^{\mathbb{Z}}$. In this paper, only the case where Γ is finite is considered. A nonuniform CA with a single local rule is called a *cellular automaton*. Just like nonuniform CA characterize the continuous functions, cellular automata are exactly the continuous maps that commute with σ. See Proposition 14 for a dynamical description of nonuniform cellular automata over finitely many rules.

A *symbol projection* is a function $\pi : \Delta \to \Gamma$ between two finite alphabets Δ and Γ. Such functions naturally extend to sequences $\delta \in \Delta^{\mathbb{Z}}$ by $\pi(\delta)_i = \pi(\delta_i)$, and we use the same notation π for the extended function. We often write $\pi(\delta)$ as $\pi\delta$ when this is clearer. A bijective symbol projection is a simple example of a conjugacy: in general, a *conjugacy* between two subshifts is a shift-commuting homeomorphism from a subshift onto another. Similarly, every surjective symbol projection is an example of a factor map: in general, a *factor map* is a continuous shift-commuting surjection from one subshift onto another. Both conjugacies and factor maps are always given by a block map (that is, a cellular automaton whose domain is not necessarily equal to its codomain) [3].

A cellular automaton is said to be *permutive* in a coordinate if the local rule is a permutation in that coordinate when values of the other coordinates are fixed. It is *bipermutive* if it is permutive in the left- and rightmost coordinates. Symbol permutations are considered bipermutive in this article, and in particular, the identity map is considered bipermutive.

We define $\mathcal{S}_n$ as the class of subsets $X \subset \Delta^{\mathbb{Z}}$ such that for a suitable symbol projection $\pi : \Delta \to \Gamma$ to a finite set of local rules Γ over Σ, and for $\Sigma = \{0, \ldots, n-1\}$, the global function of (the nonuniform CA) $\pi\delta$ is surjective on $\Sigma^{\mathbb{Z}}$ if and only if $\delta \in X$. We define $\mathcal{I}_n$ analogously, by requiring that $\pi\delta$ be injective. The subsets in $\mathcal{S}_n$ are automatically subshifts [2]. Of course, the exact n symbols in Σ do not matter. We define $\mathcal{S} = \bigcup_n \mathcal{S}_n$ and $\mathcal{I} = \bigcup_n \mathcal{I}_n$. In [2], it is proved that $X \in \mathcal{S}$ is always sofic, and that $X \in \mathcal{I}$ is ζ-automatic, that is, it is accepted by a Büchi automaton. We will not need the latter fact, and thus omit the definition of a ζ-automatic set.

In the context of realization problems for nonuniform CA, we use subshifts in three different roles. To perhaps prevent some confusion, let us clarify the situation and choose some naming conventions.

- The full shift $\Sigma^{\mathbb{Z}}$ is the subshift on which our nonuniform CA act, and we usually call Σ the set of *states*. Sequences in $\Sigma^{\mathbb{Z}}$ are called *input points*, and we use variables in the x, y, z family for them.

- We have a subshift $X \subset \Delta^{\mathbb{Z}}$ (or $Y \subset \Theta^{\mathbb{Z}}$) which we are trying to realize (or assume to be realized), through a projection to a set of local rules, as the set of surjective or injective nonuniform CA. We use the variable δ and its variants for points of $\Delta^{\mathbb{Z}}$.

- Implicitly, we use the subshift of nonuniform CA obtained through a projection π from Δ to a set of local rules Γ. We usually only use sequences of such a subshift through π by $\pi\delta$ where $\delta \in \Delta^{\mathbb{Z}}$.

We sometimes say that the subshift $X \subset \Delta^{\mathbb{Z}}$ has *realization* (π, Σ), and the interpretation is that $\pi : \Delta \to \Gamma$ is a projection onto local rules on the alphabet Σ, $X \subset \Delta^{\mathbb{Z}}$, and X is exactly the set of sequences $\delta \in \Delta^{\mathbb{Z}}$ such that $\pi\delta : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ has the property being discussed (usually surjectivity). We also say that X is realizable through the property, and sometimes that the property realizes X. When Σ is clear from context, we sometimes drop it, and call π a realization. When Δ is immaterial, or we are discussing nonuniform CA directly (not in the context of realizing subshifts), we may also drop the projection, and call a tuple of local maps a realization.

We use the numbering scheme of [4] of elementary cellular automata in some examples.

3 Surjectivity and Injectivity

In this section, we study the classes $\mathcal{S}$ and $\mathcal{I}$. We do not know any examples of sofic shifts not in $\mathcal{S}$, so for $\mathcal{S}$, we only show examples of sofic shifts it contains. For $\mathcal{I}$, we give a complete characterization.

We begin with some closure properties and relations between the classes $\mathcal{S}_n$. First of all, trivially, $\mathcal{S}$ is closed under symbol permutations (by changing the projection π). The following lemma follows almost as easily.

Lemma 1. *The class $\mathcal{S}$ is closed under intersection.*

Proof. Let $X \subset \Delta^{\mathbb{Z}}$ be in $\mathcal{S}_m$ with the realization π, and $Y \subset \Delta^{\mathbb{Z}}$ in $\mathcal{S}_n$ with the realization π'. We let $\pi'' : \Delta \to \Gamma$ be the map $d \mapsto (\pi d, \pi' d)$, where $(\pi d, \pi d')$ is the local rule over $\Sigma = \{0, \dots, m-1\} \times \{0, \dots, n-1\}$ that behaves as πd on the $\{0, \dots, m-1\}$ component and as $\pi' d$ on the $\{0, \dots, n-1\}$ component. It is then easy to see that $\pi'' \delta$ is surjective for $\delta \in \Delta^{\mathbb{Z}}$ if and only if both $\pi\delta$ and $\pi'\delta$ are surjective, and π'' thus realizes the subshift $X \cap Y$. $\square$

In fact, we see from the proof that $X \in \mathcal{S}_m, Y \in \mathcal{S}_n$ implies $X \cap Y \in \mathcal{S}_{mn}$. We do not know, however, whether $\mathcal{S}_n$ is closed under intersection for a fixed $n > 1$.

A similar trick does not seem to work for unions. We will see in Corollary 1 that $\mathcal{S}$ is at least closed under *symbol disjoint* union. However, even though, in a sense, a disjoint union is the same thing as a symbol disjoint union 'up to conjugacy', we do not know whether $\mathcal{S}$ is closed under disjoint union.

There is an obvious connection between the classes $(\mathcal{S}_i)_{i \in \mathbb{N}}$: the map $n \mapsto \mathcal{S}_n$ is an order preserving function from $(\mathbb{N}, |)$ to $((\mathcal{S}_i)_{i \in \mathbb{N}}, \subset)$ (assuming we restrict the class of subshifts in $\mathcal{S}$ to a set).

Lemma 2. *If $m|n$, then $\mathcal{S}_m \subset \mathcal{S}_n$.*

Proof. By letting $\Sigma = \Sigma_1 \times \Sigma_2$, where $|\Sigma_1| = m$ and $|\Sigma_2| = \frac{n}{m}$, we can have all the local maps preserve the Σ_2 component and operate as the appropriate local map on the Σ_1-components of cells. This way, all subshifts in $\mathcal{S}_m$ can be realized in $\mathcal{S}_n$. $\square$

We do not know whether the sets $\mathcal{S}_n$ are distinct for different $n > 1$, so the inclusions may not be proper.

Definition 1. *Let Δ and Θ be alphabets, and let $g : \Delta \mapsto \Theta^k$ be a function for some k. For $\delta \in \Delta^{\mathbb{Z}}$, we define*

$$g(\delta) = \cdots g(\delta_{-2})g(\delta_{-1}).g(\delta_0)g(\delta_1)\cdots \in \Theta^{\mathbb{Z}},$$

concatenating the words $g(\delta_i) \in \Theta^k$ together. For a subshift $Y \subset \Theta^{\mathbb{Z}}$, the uniform desubstitution *of Y by g is the subshift $X = \{\delta \in \Delta^{\mathbb{Z}} \mid g(\delta) \in Y\}$.*

Lemma 3. *The class $\mathcal{S}$ is closed under uniform desubstitution.*

Proof. Let $g : \Delta \mapsto \Theta^k$ be arbitrary, and suppose $Y \subset \Theta^{\mathbb{Z}}$ is in $\mathcal{S}$ with realization $(\pi', \bar{\Sigma})$, such that all the local rules $\pi'\Theta$ have neighborhood $[-r, r]$. For a word $u \in \Theta^k$, define the function $\pi' u : \bar{\Sigma}^{k+2r} \to \bar{\Sigma}^k$ in the obvious way, by applying the local rule πu_i in the coordinate w_{i+r} of a given word $w \in \bar{\Sigma}^{k+2r}$, for $i \in [1, k]$. For $s \in \Delta$, let πs be the local rule (P_s, Σ, f_s), where $P_s = [-r, r]$ (this is overkill if $k > 1$), $\Sigma = \bar{\Sigma}^k$, and

$$f_s(w) = \pi'(g(s))(w_{[rk+1-r,(r+1)k+r]}),$$

where the word $w \in \Sigma^{2r+1} = (\bar{\Sigma}^k)^{2r+1}$ is thought of as a word in $\bar{\Sigma}^{k(2r+1)}$ by concatenating the $2r+1$ words $w_i \in \bar{\Sigma}^k$. Note that our indexing of words starts at 1.

For $x \in \Sigma^{\mathbb{Z}}$, define $\phi(x) = \cdots x_{-2}x_{-1}.x_0x_1\cdots \in \bar{\Sigma}^{\mathbb{Z}}$ by concatenating the words $x_i \in \Sigma = \bar{\Sigma}^k$ together, and observe that this is a bijection between $\Sigma^{\mathbb{Z}}$ and $\bar{\Sigma}^{\mathbb{Z}}$. Now, for $\delta \in \Delta^{\mathbb{Z}}$, consider the nonuniform CA $\pi\delta$. By the definition of the local rules, we have

$$\pi\delta(x) = \pi' g(\delta)(\phi(x))$$

for all $x \in \Sigma^{\mathbb{Z}}$. In particular, $\pi\delta$ is surjective if and only if $\pi' g(\delta)$ is surjective if and only if $g(\delta) \in Y$, so that π is a realization of the uniform desubstitution of Y by g. □

Now, let us proceed to our main results: SFTs are in $\mathcal{S} \cap \mathcal{I}$ and they are the only subshifts in $\mathcal{I}$.

Definition 2. *Let $\mathcal{F}$ be any field, and $n \geq 2$. We write I_n for the identity matrix of dimensions $n \times n$, and J_n for the matrix*

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ a & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

also of dimensions $n \times n$, where $a = (-1)^n$. This is the sum of the identity matrix with its rotated version, with the bottom left value possibly negated depending on the parity of n.

Definition 3. *Let $n \geq 2$ and let u_i, v_i for $i \in [1, n]$ be row vectors of the same length, and let (M_1, M_2) be a pair of matrices, such that* $M_1 = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ *and* $M_2 = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. *For $w \in \{0,1\}^n$, we write $[M_1, M_2]_w$ for the matrix* $\begin{pmatrix} [u_1, v_1]_{w_1} \\ \vdots \\ [u_n, v_n]_{w_n} \end{pmatrix}$, *where we write $[a, b]_0 = a$ and $[a, b]_1 = b$.*

Lemma 4. *If $n \geq 2$, then $\det[J_n, I_n]_w = 0$ if and only if $w = 0^n$ (with any underlying field).*

Proof. If $w \neq 0^n$, then when computing the determinant by summing signed permutations, only the identity permutation appears with nonzero coefficient, and $\det[J_n, I_n]_w = 1$. On the other hand, clearly $\det[J_n, I_n]_{0^n} = \det J_n = 0$, since only two permutations with nonzero coefficients appear and their signs cancel each other, as the coefficient for the cycle $(1, \ldots, n)$ is

$$\mathrm{sign}((1, \ldots, n)) \cdot a = (-1)^{n-1} \cdot (-1)^n = -1.$$

□

To illustrate the usefulness of Lemma 4, consider the map $v \mapsto ([J_n, I_n]_w v^T)^T$, where $v \in \mathbb{Z}_2^n$ (seen as a row vector) and $n \geq 2$. Since $\mathbb{Z}_2^n$ is finite, this map is surjective (equivalently, injective or periodic) if and only if $[J_n, I_n]_w$ has nonzero determinant. We can interpret this as a statement about nonuniform cellular automata with 'periodic boundary conditions'. Namely, if we glue the left and right end of each vector $v \in \mathbb{Z}_2^n$ together so that the right neighbor of the nth cell is the first cell, then the mapping $v \mapsto ([J_n, I_n]_w v^T)^T$ is realized by the nonuniform cellular automaton where the ith coordinate contains the local rule that XORs the current cell with its right neighbor if $w_i = 0$, and the identity rule otherwise. The lemma states that such a nonuniform CA is surjective (equivalently, injective or periodic), if and only if at least one of the coordinates uses the identity rule.

This is of course not hard to show directly either: If all the coordinates use the XOR rule, then 1^n and 0^n both map to 0^n, so the rule is not injective, and thus not surjective either. If, on the other hand, at least one rule uses the identity map, then one can inductively infer a unique preimage for each vector, coordinate by coordinate, starting at the coordinates using the identity map, so the map must be injective, and thus surjective. This basic idea is well-known at least in the binary case, and the reader can spot it, for example, in the second-to-last paragraph of the proof of Theorem 1 in [5].

Theorem 1. *For all m, every SFT $X \subset \Delta^{\mathbb{Z}}$ is in $\mathcal{S}_m \cap \mathcal{I}_m$.*

Proof. We prove the slightly stronger result that there exists a symbol projection $\pi : \Delta \to \Gamma$, where Γ is a set of local rules over $\Sigma = \{0, \ldots, m-1\}$, such that the following are equivalent for $\delta \in \Delta^{\mathbb{Z}}$:

- $\delta \in X$,
- the nonuniform CA $\pi\delta : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is surjective,
- the nonuniform CA $\pi\delta$ is injective, and
- the nonuniform CA $\pi\delta$ satisfies $\exists p > 0 : \forall x \in \Sigma^{\mathbb{Z}} : (\pi\delta)^p(x) = x$.

Now, let us assume m is a prime number, and let $\mathbb{F}_m$ be the finite field of order m. We explain the local rules πd for $d \in \Delta$ by specifying how they differ from the identity function.

Let $X \subset \Delta^{\mathbb{Z}}$ be defined by the finite set of forbidden words F. We may assume $|u| = n$ for all $u \in F$ by extending the shorter forbidden words in all possible ways. For each word $u \in F$, we choose a word $B(u) = 01^k0$ such that all the $B(u)$ are distinct and $k > n$. Now, in any input the occurrences of words of the form $B(u)vB(u)$ where $|v| = n$ are disjoint in their v-parts, and a local map can easily detect if such a pattern occurs around the cell it is rewriting.

The idea is to construct the local maps in such a way that only words of the form $B(u)vB(u)$ are changed, and only in their v-part, and the image does not depend on anything outside the word. Then, a nonuniform CA is surjective (equivalently injective or periodic) if and only if for every u, the block of n local maps mapping the v-part of $B(u)vB(u)$ gives a bijective map.

We obtain these mappings from J_n and I_n (taken over the field $\mathbb{F}_m$) as follows: If the local map πd finds itself rewriting the ith coordinate of the subword $v \in \Sigma^n$ in an occurrence $B(u)vB(u)$ in the input point $x \in \Sigma^{\mathbb{Z}}$, it checks whether $u_i = d$. If this is the case, πd outputs $(J_n v^T)_i$, and otherwise it outputs $(I_n v^T)_i$.

Now, consider the mapping among words of the form $B(u)vB(u)$ for $u \in F$ when a block of local maps $\pi d_1, \ldots, \pi d_n$ is rewriting v. Let $w \in \{0,1\}^n$ be defined by $w_i = 0$ if $d_i = u_i$, and $w_i = 1$ otherwise. Then, $B(u)vB(u)$ is rewritten to $B(u)([J_n, I_n]_w v^T)^T B(u)$, so the mapping is not bijective if and only if $w = 0^n$ by Lemma 4.

So, consider a sequence $\delta \in \Delta^{\mathbb{Z}}$. If $\delta_{[j,j+n-1]} = u$ for some $u \in F$, then $\pi\delta$ is not surjective, injective or periodic, by considering input points x where $x_{[j-|B(u)|,j+n-1+|B(u)|]} = B(u)vB(u)$. On the other hand, if no forbidden word occurs, we clearly have $\forall x : \pi\delta^p(x) = x$ for some p. Namely, all rewritings happen in patterns of the form $B(u)vB(u)$ with $|v| = n$, and the patterns $B(u)$ for $u \in F$ are never moved, removed or introduced. As the mapping on the v-part is bijective, it is periodic and there exists a period $p(u)$ for every forbidden pattern u. As the number of forbidden words is finite, $p = \mathrm{lcm}_u p(u)$ is a global period for $\pi(\delta)$.

In the case when m is not prime, we simply separate a track of prime alphabet size p as in the proof of Lemma 2 and construct the local rules as above for this track, while behaving as identity on the other track (of alphabet size $\frac{m}{p}$). □

We note that Theorem 1 is completely algorithmic: the proof provides a procedure for constructing local rules realizing a given SFT. The radius of each of the local rules is linear in the maximal length of a forbidden word, and the number of forbidden words (and can be made more efficient by choosing the unbordered words $B(u)$ in a smarter way).

We say $X \subset \Delta^{\mathbb{Z}}$ and $Y \subset \Theta^{\mathbb{Z}}$ are symbol disjoint if $\Delta \cap \Theta = \emptyset$, and in this case their union $X \cup Y \subset (\Delta \cup \Theta)^{\mathbb{Z}}$ is called their *symbol disjoint union*.

Corollary 1. *The class $\mathcal{S}$ is closed under symbol disjoint union.*

Proof. Let $X \subset \Delta^{\mathbb{Z}}$ and $Y \subset \Theta^{\mathbb{Z}}$ be in $\mathcal{S}$ with $\Delta \cap \Theta = \emptyset$. By Lemma 2, they are in fact both in $\mathcal{S}_n$ for some n by projections $\pi : \Delta \to \Gamma$ and $\pi' : \Theta \to \Gamma'$, respectively.

Consider the symbol projection $\pi'' : (\Delta \cup \Theta) \to (\Gamma \cup \Gamma')$, the piecewise map equal to π on Δ and equal to π' on Θ. Obviously, if $\delta \in (\Delta \cup \Theta)^{\mathbb{Z}}$ only contains symbols from Δ (resp. Θ), $\pi''\delta$ is surjective if and only if the nonuniform CA $\pi\delta$ (resp. $\pi'\delta$) is surjective. Thus, we have proved that Z is in $\mathcal{S}$ for some Z such that $Z \cap \Delta^{\mathbb{Z}} = X$ and $Z \cap \Theta^{\mathbb{Z}} = Y$. But this means also the subshift $Z \cap (\Delta^{\mathbb{Z}} \cup \Theta^{\mathbb{Z}}) = X \cup Y$ is in $\mathcal{S}$ by Theorem 1 and Lemma 1, since $(\Delta^{\mathbb{Z}} \cup \Theta^{\mathbb{Z}})$ is an SFT. □

Theorem 2. *The closed sets in $\mathcal{I}$ are exactly the SFTs.*

It is shown in [2] that the complements of subshifts in $\mathcal{I}$ are accepted by automata with the 'quick-fail accepting condition', and it is not hard to show that all closed sets of sequences accepted by such automata are in fact SFTs. However, we give the following direct proof instead.

Proof. SFTs are closed sets, and they are in $\mathcal{I}$ by Theorem 1. For the other inclusion, let $X \subset \Delta^{\mathbb{Z}}$ be a closed set in $\mathcal{I}_n$ realized by the surjective projection $\pi : \Delta \to \Gamma$ where Γ is a finite set of local maps over $\Sigma = \{0, \dots, n-1\}$. Since every set in $\mathcal{I}$ is clearly shift-invariant, X is a subshift.

We first show that there exists r such that if $\delta \in X$ and $x_0 \neq y_0$ for $x, y \in \Sigma^{\mathbb{Z}}$, then $(\pi\delta)(x)_i \neq (\pi\delta)(y)_i$ for some $i \in [-r, r]$: If this were not the case, we would find, for each r, a triple (δ^r, x^r, y^r) such that $x^r_0 \neq y^r_0$ but $(\pi\delta^r)(x^r)_{[-r,r]} = (\pi\delta^r)(y^r)_{[-r,r]}$. Now, let (δ, x, y) be any limit point of the sequence $(\delta^r, x^r, y^r)_{r \in \mathbb{N}}$ in the subshift $(\Delta \times \Sigma \times \Sigma)^{\mathbb{Z}}$. Then, we easily see that $x_0 \neq y_0$ but $(\pi\delta)(x) = (\pi\delta)(y)$, and thus $\pi\delta$ is not injective. This is a contradiction because X was assumed to be closed and πX to be exactly the set of injective nonuniform CA over Γ.

Now, consider two words $uv, vw \sqsubset X$ such that $|v| \geq 2r + 1$. We claim that $uvw \sqsubset X$, which then implies that X is an SFT; this is in fact a well-known characterization of SFTs [3]. Take an extension $z_\ell u.vz_r \in X$ for uv where the decimal point indicates the origin to be on its right, and an extension $z'_\ell.vwz'_r \in X$ for vw. Now, if $z_\ell u.vwz'_r$ were not injective through π, there would exist inputs $x, y \in \Sigma^{\mathbb{Z}}$ such that $x \neq y$ but $\pi(z_\ell u.vwz'_r)(x) = \pi(z_\ell u.vwz'_r)(y)$.

Let i be such that $x_i \neq y_i$. Since $|v| \geq 2r + 1$, there must be at least r coordinates of v either to the left or to the right of i in $z_\ell u.vwz'_r$. By symmetry, we may assume they are to the left of i, that is $i \geq r$. But then, since

$\pi(z'_\ell.vwz'_r)(x)_{[i-r,i+r]} \neq \pi(z'_\ell.vwz'_r)(y)_{[i-r,i+r]}$, we must have

$$\pi(z_\ell u.vwz'_r)(x)_{[i-r,i+r]} \neq \pi(z_\ell u.vwz'_r)(y)_{[i-r,i+r]},$$

as the local maps computing the cells $[i-r, i+r]$ are unchanged, which is a contradiction. Therefore, $uvw \sqsubset X$, and thus X is an SFT. □

Note that from the previous theorems, it follows that the SFTs are exactly the sets of sequences which are simultaneously the set of injective and surjective sequences for some set of local rules.

It is known that, contrary to the case of $\mathcal{I}$, subshifts in $\mathcal{S}$ do not coincide with SFTs, but are a proper superclass. The properness was already shown in Example 11 of [2], where the following example was given (and which we reprove in Example 1):

Proposition 1. *The (proper sofic)* even shift $\mathcal{B}^{-1}((1(00)^*)^*)$ *is in* $\mathcal{S}_2$*.*

Here, the proper soficness is due to doing modular counting of a subpattern (the parity of the run of 0s): the forbidden patterns of an SFT cannot 'see' the parity of a long enough run of 0s. Another type of counting a sofic shift may perform is counting *up to* a threshold. We show by example that a subshift in $\mathcal{S}_2$ can at least count to 1 and 2:

Proposition 2. *The subshifts* $\mathcal{B}^{-1}(0^*10^*)$ *and* $\mathcal{B}^{-1}(0^*10^*10^*)$ *are in* $\mathcal{S}_2$*.*

Proof. Using the algorithm in [2], one can check that (among many other pairs) the elementary cellular automata 90 and 2 realize the first subshift, and the elementary cellular automata 150 and 116 realize the second. □

With a larger alphabet, we can even count to n.

Proposition 3. *The subshift* $X = \mathcal{B}^{-1}((0^*1)^{n-1}0^*)$ *is in* $\mathcal{S}_n$*.*

Proof. The local rules $\pi 0$ and $\pi 1$ are over the alphabet $\mathbb{Z}_n$ (with its usual abelian group structure) and both have neighborhood $\{-1, 0\}$. For $(a, b) \neq (0, 1)$, we let $\pi 0(a, b) = \pi 1(a, b) = b - a$, and we let $\pi 0(0, 1) = 1$ and $\pi 1(0, 1) = 0$. Thus, $\pi 0$ computes the difference of its neighbors, and $\pi 1$ computes the difference, except for mapping one input wrong.

First, we produce the De Bruijn graph as in [2], with states $\mathbb{Z}_n$, and labeled edges

$$\{(a, b, (c, b-a)) \mid (a, b) \neq (0, 1), c \in \{0, 1\}\} \cup \{(0, 1, (0, 1)), (0, 1, (1, 0))\},$$

where the third component denotes the label. The label is (d, s) where πd is the local rule being applied and $s \in \Sigma$ is the output symbol. This graph is thought of as a nondeterministic finite state automaton with all states initial and final. The language it accepts consists of words $(u, v) \in (\{0, 1\} \times \mathbb{Z}_n)^*$ such that the sequence πu of local rules maps some word of length $|v| + 1$ over $\mathbb{Z}_n$ onto v.

Denote by $G = (V, E)$ the DFA obtained by the subset construction for the complement of the language of the NFA defined above, having the single initial state $\mathbb{Z}_n$ and the single final state $\emptyset$. By erasing the second component of each label, one would obtain an NFA for the forbidden words of the subshift realized by $\pi 0$ and $\pi 1$ (elaborated in [2]), so that the forbidden words are exactly the words w such that there exists some $u \in \mathbb{Z}_n^{|w|}$ for which the path in G with label $(w, u) = (w_1, u_1), (w_2, u_2), \ldots, (w_{|w|}, u_{|w|})$ from the initial state $\mathbb{Z}_n$ leads to the state $\emptyset$.

We need to show that $w = 0^{k_0} 10^{k_1} 10^{k_2} 1 \ldots 10^{k_m} 10^{k_{m+1}}$ for $k_i \geq 0$ is forbidden if and only if $m \geq n-1$. First, by induction, it is easy to see that if $m < n-1$, then w is not forbidden: if $A \xrightarrow{(0,a)} B$ for $A, B \in V$, then $|B| = |A|$, and if $A \xrightarrow{(1,a)} B$, then $|B| \geq |A| - 1$, so the empty set cannot be reached from the state $\mathbb{Z}_n$ after reading a word where less than n coordinates contain 1.

To show that w *is* forbidden if $m \geq n-1$, we simply need to find u such that $\mathbb{Z}_n \xrightarrow{(w,u)} \emptyset$. We show this even in the case $k_0 = k_{m+1} = 0$:

$$u = w = 10^{k_1} 10^{k_2} 1 \ldots 10^{k_m} 1$$

has this property, since after reading reading the first ℓ occurrences of $(1, 1)$, G is in state $\{\ell + 1, \ldots, n-1, 0\}$. □

Using Lemma 3, Theorem 1 and Lemma 1, one can also count more complicated patterns. For example, to obtain the subshift $\mathcal{B}^{-1}((0^*12)^n 0^*)$, one can take the preimage of the subshift $\mathcal{B}^{-1}((0^*1)^{2n} 0^*)$ in the substitution $0 \mapsto 0; 1 \mapsto 1; 2 \mapsto 1$, and intersect the result with the SFT $\mathcal{B}^{-1}((0 + 12)^*)$.

We now give the simplest candidate sofic shifts which we haven't been able to realize; they are indeed very simple:

Question 2. *Is any the following sofic shifts in $\mathcal{S}$?*

- $X_1 = \mathcal{B}^{-1}(0^*1^* + 1^*0^*)$
- $X_2 = \mathcal{B}^{-1}((00 + 01)^*)$
- $X_3 = \mathcal{B}^{-1}(0^*1^*0^*)$

The subshift X_2 could be called the 'odd shift', as it is the subshift where the number of 0s between two successive 1s is odd. Note that the even shift, where the runs of 0s are even, is in $\mathcal{S}$ by Proposition 1. If X_3 is realizable, then X_1 is as well, as $X_1 = X_3 \cap \text{flip}(X_3)$, where flip is the symbol permutation $0 \mapsto 1, 1 \mapsto 0$.

We have checked that at least no pair of elementary CA realizes X_1, X_2 or X_3. All surjective two-neighbor automata on prime alphabets are permutive in at least one coordinate [6], so the proof of Lemma 8 shows that a pair of such maps cannot realize X_1 or X_3. A computer search shows that X_2 is at least not realized by a pair of two-neighbor three-state automata. In the following sections, we will show other very restricted classes of automata which cannot realize these subshifts. Namely, we show that bipermutive and asynchronous automata cannot realize X_1 and X_2, and that certain types of rules with algebraic structure cannot realize any of the three.

Note that if $\mathcal{S}$ were closed under factor maps, all sofic shifts would be in $\mathcal{S}$ by Theorem 1 and the well-known fact that sofic shifts are exactly the closure of SFTs under factor maps [3]. Since Theorem 1 in a sense shows that local rules can locally detect which local rules neighbor them, it makes sense to ask if $\mathcal{S}$ is closed under conjugacy. We do not know whether this is the case.

Question 3. *Is the class $\mathcal{S}$ closed under conjugacy?*

4 Asynchronous Cellular Automata

Nonuniform cellular automata relax the definition of cellular automata by allowing a different rule in each cell. Other (usually orthogonal) generalizations are obtained by varying the order in which the cells update their state. This is often referred to as the study of asynchronous cellular automata. There are many ways to formalize this basic idea, and some of the various approaches are surveyed in [7]. Our definition is rather different from the usual ones, in that we define an asynchronous CA as a nonuniform CA with two block maps, of which one is the identity map. Of course, for any given nonuniform CA over such a pair of local rules, the dynamics is not really 'asynchronous' in any way. However, asking *realizability questions* for such a pair of local rules can be considered a way to study the asynchronous behavior of a CA, since we are asking which subsets $N \subset \mathbb{Z}$ of the cells are such that updating the CA in cells N, but not the cells $\mathbb{Z} \setminus N$, updates the global configuration surjectively (or injectively).

We note that there are certainly also more intrinsic realizability questions for asynchronous CA. For a particular rule, we can, for example, ask which updating orders for the cells lead to surjective maps or injective maps. One way to encode such orders as words over $\{<, \equiv, >\}$ is used in [8] in the context of asynchronous CA on finite words, and this can be extended to the infinite case as well (although care needs to be taken to find the right questions to ask, and additional technical problems arise in the infinite case). We do not study such questions here, but suggest this as a possible topic for future research.

Now, consider a pair (F, G) of local rules, with F the identity map. We are interested in the subshift realized through surjectivity or injectivity, with such a pair. Of course, the realized subshifts are still sofic, and obvious further restrictions are that the subshift is binary (since there are two local rules) and that it contains the unary point (since the identity CA is surjective and injective). We can show that all binary SFTs containing a unary point can indeed be realized as surjective sequences, but not all such sofic shifts can be realized. For injectivity, we again exactly characterize the subshifts realized. Restricting to asynchronous automata, we write $\mathcal{S}_A$ and $\mathcal{I}_A$ for sets that can be realized as surjective or injective sequences, respectively.

We begin with a simple observation that lets us replace any reversible CA by the identity map in a realization. If $F : \Sigma^{[-m,m]} \to \Sigma$ and $G : \Sigma^{[-n,n]} \to \Sigma$ are local rules, we write

$$(G \circ F)(a_{-n-m}, \ldots, a_{n+m}) = G(F(a_{-n-m}, \ldots, a_{-n+m}), \ldots, F(a_{n-m}, \ldots, a_{n+m})).$$

Writing $f = {}^{\infty}F^{\infty}$ and $g = {}^{\infty}G^{\infty}$, we have $({}^{\infty}(G \circ F)^{\infty})(x) = (g \circ f)(x))$. From this, and the fact that an injective cellular automaton is surjective, we obtain the following lemma.

Lemma 5. *Let $\pi : \Delta \to \Gamma$ be a map onto local rules Γ on Σ, and take a common neighborhood $[-n, n]$ for all the πd, where $d \in \Delta$. Suppose that f is a surjective cellular automaton on $\Sigma^{\mathbb{Z}}$ with the local map $F : \Sigma^{[-m,m]} \to \Sigma$. Then $\pi' : \Delta \to \Gamma'$ defined by $\pi' d = \pi d \circ F$ has the same surjective sequences as π. If f is injective, then π' and π also have the same injective sequences.*

The lemma states that we may compose a surjective (resp. injective) cellular automaton f before each of the rules πd, and the subshift realized through surjectivity (resp. injectivity) does not change. Note that if, say, $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is injective, then of course πd is injective if and only if $f \circ \pi d$ is. However, there is no direct analogue of Lemma 5 for compositions on the left, since while $\pi d \mapsto \pi d \circ f$ is a matter of changing the projection π, the map $\pi d \mapsto f \circ \pi d$ is not.

For conciseness, in this section we write $\mathcal{U}$ for subshifts that have at least one unary point and $\mathcal{N}_n$ for subshifts over an alphabet of size n. We also use SFT and sofic in formulas to stand for the corresponding classes.

Proposition 4.

$$\mathcal{I}_A = \mathcal{I} \cap \mathcal{U} \cap \mathcal{N}_2 (= \mathrm{SFT} \cap \mathcal{U} \cap \mathcal{N}_2)$$

and

$$\mathrm{SFT} \cap \mathcal{U} \cap \mathcal{N}_2 \subset \mathcal{S}_A \subset \mathcal{S} \cap \mathcal{U} \cap \mathcal{N}_2$$

Proof. The inclusions $\mathcal{I}_A \subset \mathcal{I} \cap \mathcal{U} \cap \mathcal{N}_2$ and $\mathcal{S}_A \subset \mathcal{S} \cap \mathcal{U} \cap \mathcal{N}_2$ follow from the definitions and the fact that the identity map is surjective and injective.

For the inclusion $\mathcal{I} \cap \mathcal{U} \cap \mathcal{N}_2 \subset \mathcal{I}_A$, suppose $X \in \mathcal{I} \cap \mathcal{U} \cap \mathcal{N}_2$, and let π with $\pi 0 = F$ and $\pi 1 = G$ be a realization. Then, since there is a unary point in X, one of the cellular automata ${}^{\infty}F^{\infty}$ and ${}^{\infty}G^{\infty}$ is injective, say ${}^{\infty}F^{\infty}$ is. An injective CA is reversible, so we let F^{-1} be the local rule of the inverse of ${}^{\infty}F^{\infty}$. Now, we note that a sequence over $\{F \circ F^{-1}, G \circ F^{-1}\}$ is injective is and only if the corresponding sequence over $\{F, G\}$ is, by Lemma 5. Since one of the local rules is the identity map, $X \in \mathcal{I}_A$ by the realization $\pi'(0) = \mathrm{id}$, $\pi'(1) = G \circ F^{-1}$. The inclusion $\mathrm{SFT} \cap \mathcal{U} \cap \mathcal{N}_2 \subset \mathcal{S}_A$ holds by a similar trick, and the fact that sequences given by the proof of Theorem 1 are injective exactly when they are surjective. ☐

Proposition 5. *The subshift X_1 of Question 2 is not in $\mathcal{S}_A$. In particular,*

$$\mathrm{SFT} \cap \mathcal{U} \cap \mathcal{N}_2 \subsetneq \mathcal{S}_A \subsetneq \mathrm{sofic} \cap \mathcal{U} \cap \mathcal{N}_2$$

Proof. One of the local rules realizing the subshift in Proposition 1 (or Example 1 in the next section) is the identity map, so $\mathcal{S}_A$ contains a proper sofic shift.

On the other hand, we show that it does not contain the subshift X_1 of Question 2. Suppose that it does, by a projection π. Then, because the subshift is symmetric, we may assume $\pi 1$ is the identity map. Let $[-r, r]$ be the neighborhood of $\pi 0$. Now, from the surjectivity of $f = \pi({}^\infty 0.1^\infty)$ and the surjectivity of $g = \pi({}^\infty 1.1^r 0^\infty)$, it follows that also $\pi({}^\infty 0.1^r 0^\infty)$ is surjective. Namely, for the preimages y and y' of x for f and g, respectively, we have $y_{[0,r-1]} = y'_{[0,r-1]} = x_{[0,r-1]}$, so that the left tail of y and the right tail of y' can be glued together to form a preimage of x for the map $\pi({}^\infty 0.1^r 0^\infty)$. Thus, π realizes a subshift larger than X. □

One can similarly show that X_2 is not in $\mathcal{S}_A$: In a realization of X_2, $\pi 0$ must be the identity map since ${}^\infty 1^\infty \notin X_2$, and one can carry out a gluing argument similar to the previous proof.

We end this section with the following connection between realizability questions for $\mathcal{S}$ and $\mathcal{S}_A$. In particular, this applies to X_1 and X_2 in Question 2.

Proposition 6. *If a subshift X is not in $\mathcal{S}_A$, but is in $\mathcal{S}$ with local rules (F, G), then neither of the rules ${}^\infty F^\infty$ or ${}^\infty G^\infty$ is injective.*

Proof. Suppose that, say, ${}^\infty F^\infty$ is injective. As in the proof of Proposition 4, using Lemma 5 we may replace the local rule F by the identity map $F \circ F^{-1}$, and G by $G \circ F^{-1}$, without affecting the surjective sequences. But then X is in $\mathcal{S}_A$, a contradiction. □

5 Nonuniform CA from Group Homomorphisms

In this section, we consider surjectivity of sequences over rules which are group homomorphisms, and also discuss bipermutivity as a technical tool. Considering homomorphic, linear or bipermutive rules is interesting because the local rules realizing the even shift [2] are linear, and thus bipermutive, for the group $\mathbb{Z}_2$, and because the problem and especially its dual statement (see Definition 4) are also interesting as linear algebraic problems. We only consider realizations through surjectivity in this section.

For each finite abelian group G, we define two classes of sofic shifts, $\mathcal{H}_G$ and $\mathcal{L}_G$. The sofic shift $X \subset \Delta^{\mathbb{Z}}$ is in $\mathcal{H}_G$ if it is realized (through surjectivity) by a projection $\pi : \Delta \to \Gamma$ to a finite set Γ of local maps, such that all the local rules πd for $d \in \Delta$ have the same neighborhood N, and each map $\pi d : G^N \to G$ is a group homomorphism (where G^N is the direct product of $|N|$ copies of G). We let $\mathcal{H} = \bigcup_G \mathcal{H}_G$. We call maps of the form $(g_1, \ldots, g_{|N|}) \mapsto a_1 g_1 + \cdots + a_{|N|} g_{|N|}$ where $a_i \in \mathbb{N}$ *linear*, and we define $\mathcal{L}_G$ as the class of sofic shifts realized by linear local rules over the group G. We define $\mathcal{L} = \bigcup_G \mathcal{L}_G$. We define $\mathcal{P}_n$ as the class of sofic shifts with a realization with only bipermutive rules (possibly with different neighborhoods) over $\Sigma = \{0, \ldots, n-1\}$, and again $\mathcal{P} = \bigcup_n \mathcal{P}_n$.

Note that the assumption that all the local rules have the same neighborhood N in the definition of the classes $\mathcal{H}_G$ and $\mathcal{L}_G$ is immaterial: the neighborhood of a group homomorphism or a linear map can be increased by simply ignoring arguments (in the case of linear maps, this means choosing $a_i = 0$ for some $i \in N$). By composing the local rules with a shift, we can usually further assume $N = \{1, \dots, |N|\}$. When $G = \mathbb{Z}_p$ and p is prime, G naturally has the structure of a field (multiplication being repeated addition). In this case, we write $a * b$ for the multiplication of the field. The notation $u \cdot v$ is reserved for concatenation, or repeated addition in the case ag where $a \in \mathbb{N}$, $g \in G$. When $g \in \mathbb{Z}_p$ and $a \in [0, p-1]$, we can of course consider a an element of $\mathbb{Z}_p$ in the obvious way, and then $ag = a * g$.

Lemma 6. *For any abelian group G, $\mathcal{L}_G \subset \mathcal{H}_G$.*

Proof. Let $F : G^N \to G$ be the linear map $F(g_1, \dots, g_{|N|}) = a_1 g_1 + \cdots + a_{|N|} g_{|N|}$. Then F is a homomorphism from the group G^N to G by a direct calculation:

$$\begin{aligned} F((g_1, \dots, g_{|N|}) + (h_1, \dots, h_{|N|})) &= F(g_1 + h_1, \dots, g_{|N|} + h_{|N|}) \\ &= a_1 g_1 + a_1 h_1 + \cdots + a_{|N|} g_{|N|} + a_{|N|} h_{|N|} \\ &= F(g_1, \dots, g_{|N|}) + F(h_1, \dots, h_{|N|}) \end{aligned}$$

Thus, any realization that shows X is in $\mathcal{L}_G$ also shows that it is in $\mathcal{H}_G$. □

Lemma 7. *If p is prime, then $\mathcal{L}_{\mathbb{Z}_p} = \mathcal{H}_{\mathbb{Z}_p} \subset \mathcal{P}_p$.*

Proof. As in the previous proof, we show that this is true by the form of the local rules allowed: all linear local rules are automatically bipermutive, and linear local rules are the same as homomorphic local rules in the case of the group $\mathbb{Z}_p$.

To see $\mathcal{L}_{\mathbb{Z}_p} \subset \mathcal{P}_p$, consider any local rule $F(g_1, \dots, g_{|N|}) = a_1 g_1 + \cdots + a_{|N|} g_{|N|}$, where we assume $N = \{1, \dots, |N|\}$. If $a_i g = 0$ for all $g \in \mathbb{Z}_p$ and $i \in \{1, \dots, |N|\}$, then $F(G^N) = \{0\}$, so no sequence of local rules containing F is surjective as a nonuniform CA, and F can be removed from any realization without changing the subshift realized. Otherwise, let $\ell \in N$ be minimal such that $a_\ell g \neq 0$ for some $g \in \mathbb{Z}_p$, and symmetrically let $r \in N$ be the maximal integer with this property (where possibly $\ell = r$). Since $\mathbb{Z}_p$ has the structure of a field with multiplication corresponding to repeated addition, we have $a_\ell g = a_\ell * g$ and $a_r g = a_r * g$. Thus, the local rule is bipermutive with neighborhood $N \cap [\ell, r]$.

It is a simple exercise to show that each group homomorphism $F : \mathbb{Z}_p^N \to \mathbb{Z}_p$ is linear. Thus, $\mathcal{H}_{\mathbb{Z}_p} \subset \mathcal{L}_{\mathbb{Z}_p}$. The equality $\mathcal{L}_{\mathbb{Z}_p} = \mathcal{H}_{\mathbb{Z}_p}$ follows from this, and the previous lemma. □

Of course, by definition, we always have $\mathcal{P}_{|G|} \cup \mathcal{H}_G \subset \mathcal{S}_{|G|}$.

One could argue that that class $\mathcal{H}$ is conceptually the most natural of these. At least, Lemma 1 holds for this class, as the product of two abelian groups is abelian and the product of two endomorphisms is an endomorphism. The other two classes are not closed under intersection by Corollary 2. The class $\mathcal{H}$ is already somewhat difficult to analyse.

Question 4. *Which sofic shifts are in $\mathcal{H}_G$, for a given abelian group G?*

The main result of this section is (a technique which proves) that $\mathcal{H}_G$ does not contain X_3 in the particularly simple case that the group G is of squarefree order.

Before discussing linearity in detail, we consider bipermutivity in itself, showing a simple property that all subshifts in $\mathcal{P}$ must have.

Lemma 8. *If $X \subset \Delta^{\mathbb{Z}}$ is in $\mathcal{P}$, then for every pair of lettters $a, b \in \Delta$, either ${}^{\infty}a.b^{\infty} \in X$ or ${}^{\infty}b.a^{\infty} \in X$.*

Proof. Let π be a realization of X. If the rightmost cell of the neighborhood of πa is i and that of πb is j, then

- if $i < j$, we have ${}^{\infty}a.b^{\infty} \in X$,
- if $i > j$, we have ${}^{\infty}b.a^{\infty} \in X$, and
- if $i = j$, we have $\{a, b\}^{\mathbb{Z}} \subset X$.

In each case, one can build a preimage for any point from left to right. □

We already get the following strong non-closure results.

Corollary 2. *Not all SFTs are in $\mathcal{P}$, $\mathcal{P}$ is not closed under conjugacy, and $\mathcal{P}$ is not closed under symbol disjoint union or intersection. The same claims are true for the class $\mathcal{L}_{\mathbb{Z}_p}$ when p is prime.*

Proof. Let $G = \mathbb{Z}_p$ for prime p. By Lemma 7, we have $\mathcal{L}_G \subset \mathcal{P}$. The SFT $X = \mathcal{B}^{-1}(0^*21^*)$ is not in $\mathcal{P}$ (and thus not in $\mathcal{L}_G$), since if $\pi 2$ is bipermutive, then $\pi^{\infty}2^{\infty}$ is surjective, so π does not realize X. Since X is conjugate to $Y = \mathcal{B}^{-1}(0^*1^*)$ and Y is realized by the (bipermutive and linear) identity map and the left shift, $\mathcal{P}$ and $\mathcal{L}_G$ are not closed under conjugacy. Since $\mathcal{B}^{-1}(0^*)$ and $\mathcal{B}^{-1}(1^*)$ are (trivially) realizable by linear rules but their union $Z = \mathcal{B}^{-1}(0^*+1^*)$ does not satisfy Lemma 8, $\mathcal{P}$ and $\mathcal{L}_G$ are not closed under symbol disjoint union. Since $\mathcal{B}^{-1}(0^*1^*)$ and $\mathcal{B}^{-1}(1^*0^*)$ are realizable but their intersection is Z, $\mathcal{P}$ and $\mathcal{L}_G$ are not closed under intersection. □

Note that while the local rules given by Theorem 1 are in a sense given by matrix multiplication, they are far from linear. Indeed, by the previous corollary, that would be impossible.

Lemma 9. *The subshifts X_1 and X_2 in Question 2 are not in $\mathcal{P}$.*

Proof. Suppose on the contrary that X_1 has realization (π, Σ). Let N_0 be the neighborhood of $\pi 0$ and N_1 that of $\pi 1$. Swapping 0 and 1 if necessary, we may assume $\min N(0) \geq \min N(1)$ (note that X_1 is closed under symbol permutation). Then $\pi({}^{\infty}1.0^n1^{\infty})$ is surjective for any n: this follows directly from the facts that $\pi({}^{\infty}0.0^n1^{\infty})$ is surjective (as ${}^{\infty}0.0^n1^{\infty} \in X_1$), $\pi 1$ is left-permutive and $\min N_0 \geq \min N_1$.

For X_2, by the previous lemma, it is enough to observe ${}^{\infty}1^{\infty} \notin X_2$. □

When $|G|$ is prime, it follows that $X_1 \notin \mathcal{H}_G$. However, we do not know X_3 is not in $\mathcal{P}$, so that, for all we know, X_1 could be the intersection of two subshifts realized by bipermutive rules; in fact, we do not know any examples of sofic shifts which are not intersections of subshifts in $\mathcal{P}$, as long as all the local rules appear in a unary point (which is a trivial necessary condition). In the rest of this section, we show that X_1 and X_3 can not be realized even as the intersection of subshifts in $\mathcal{L}_G$ for $G = \mathbb{Z}_p$ and p prime, and in particular they are not in $\mathcal{H}_G$ when G is of squarefree order.

To prove this result, the following notion of a dual map is very helpful.

Definition 4. *Let p be a prime number, so that $\mathbb{Z}_p$ has the structure of a field. Fix $r \in \mathbb{N}$, let $\Delta = \mathbb{Z}_p^{[-r,r]}$, and let Γ be the set of radius r local rules over $\mathbb{Z}_p$. Let $\pi : \Delta \to \Gamma$ be the projection onto local rules defined by*

$$\pi d(g_{-r}, \dots, g_r) = d_{-r} * g_{-r} + \cdots + d_r * g_r,$$

for $d \in \Delta$. For $\delta \in \Delta^{\mathbb{Z}}$, we define its dual *$\delta^T \in \Delta^{\mathbb{Z}}$ by $(\delta_j^T)_k = (\delta_{j+k})_{-k}$.*

For example, if the local rule $(\pi\delta)_i$, say, adds the right neighbor to the current cell with multiplier a, then $(\pi\delta^T)_{i+1}$ adds the left neighbor to the current cell with multiplier a – in general, influences between cells are reversed. This is illustrated in Figure 1 in the case $G = \mathbb{Z}_2$. It should be clear that the operation is an involution of $\Delta^{\mathbb{Z}}$. We usually take duals of nonuniform cellular automata and finite sequences of local rules directly, with the obvious interpretations.

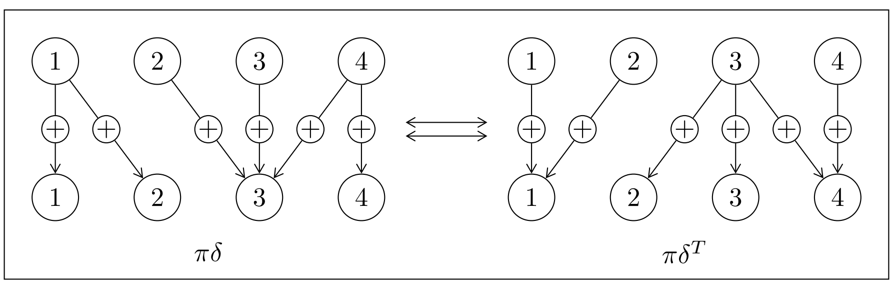

Figure 1: The dual operation. The numbered circles denote positions of the input, and a +-labeled arrow from i to j means $(\delta_j)_{i-j} = 1$.

Lemma 10 (well-known)**.** *Let M be a matrix over a field. Then $u \mapsto uM$ is surjective if and only if $v \mapsto M^T v$ is injective.*

In order to apply the previous lemma to nonuniform CA, we give a generalization of the concept of preinjectivity of cellular automata.

Definition 5. *A function $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is called* preinjective *if, whenever $x, y \in \Sigma^{\mathbb{Z}}$, $x \neq y$ and $x_i = y_i$ for large enough $|i|$, we have $f(x) \neq f(y)$.*

Intuitively, a map is preinjective if, whenever two distinct points agree in all but finitely many coordinates, their image is different. Just like in the case of cellular automata, it is not hard to show that for a nonuniform cellular automaton with finitely many local rules over Σ, preinjectivity is equivalent to injectivity over a-finite configurations for any $a \in \Sigma$.

Lemma 11. *Let $\delta \in \Delta^{\mathbb{Z}}$ and let $\pi : \Delta \to \Gamma$ be a projection onto linear local rules Γ over $\mathbb{Z}_p$ where p is prime. Then $\pi\delta$ is surjective if and only if $\pi\delta^T$ is preinjective.*

Proof. Suppose $\Delta = \mathbb{Z}_p^{[-r,r]}$ and π are as in Definition 4. For $\ell \in \mathbb{N}$, we define a $(2\ell+1) \times (2(\ell+r)+1)$ matrix M over $\mathbb{Z}_p$. To make the indexes more readable, suppose M is indexed by $[-\ell, \ell] \times [-\ell - r, \ell + r]$, and let

$$M_{i,j} = \begin{cases} (\delta_i)_{j-i}, & \text{if } j - i \in [-r, r] \\ 0, & \text{otherwise} \end{cases}$$

Now, $u \mapsto uM = (M^T u^T)^T$ for $u \in \mathbb{Z}_p^{[-\ell,\ell]}$ is injective if and only if the dual map of δ is injective among points x such that $x_k = 0$ for $k \notin [-\ell, \ell]$. Similarly, $v \mapsto Mv^T$ is surjective if and only if for all $w \in \mathbb{Z}_p^{[-\ell,\ell]}$ there exists x with $\pi\delta(x)_{[-\ell,\ell]} = w$.

It is clear that the first property holds for all ℓ if and only if the map $\pi\delta^T$ is preinjective. Since $\pi\delta(\mathbb{Z}_p^{\mathbb{Z}})$ is topologically closed, the second property holds if and only if $\pi\delta$ is surjective. Thus, Lemma 10 implies the claim. □

In fact, the proof of the previous lemma shows that a finite sequence of linear rules on $\mathbb{Z}_p$ is surjective if and only if its dual is injective. We will also need this stronger result.

For cellular automata (in particular linear ones) on the full shift, surjectivity and preinjectivity are in fact equal concepts. For linear nonuniform CA, neither implies the other: if $\pi 0$ is the identity map and $\pi 1$ the left shift, then $\pi({}^\infty 0.1^\infty)$ is surjective but not preinjective, and $\pi({}^\infty 1.0^\infty)$ is preinjective but not surjective.

The dual map provides a simple way to prove that $\pi\delta$ is not surjective: one only needs to show $\pi\delta^T$ is not preinjective, that is, to exhibit a nonzero vector that maps to zero. To prove non-surjectivity directly, one needs a case analysis that shows that some vector does not have a preimage. The following is Example 11 in [2], and it is an example of a proper sofic shift in $\mathcal{L}_{\mathbb{Z}_2}$. We illustrate the dual operation with this example.

Example 1. *We reprove Proposition 1 using the dual operation. We use the rules given in [2], so that $\pi 1$ is the identity map and $\pi 0 : \mathbb{Z}_2^{\{-1,1\}} \to \mathbb{Z}_2$ is the two-neighbor xor. These rules are linear, so that Lemma 11 applies. We need to show that $\pi\delta$ is surjective if and only if δ is in X. First, suppose δ is not in X. We may assume $\delta_{[0,2k+2]} = 10^{2k+1}1$. Then for $x = {}^\infty 0.1(10)^k 110^\infty$, where the position to the right of the decimal point is the origin, we have $\pi\delta^T(x) = {}^\infty 0^\infty$ since*

$${}^\infty 0.10^{2k+2}0^\infty + \sum_{i=0}^{k} {}^\infty 0.0^{2i}1010^{2(k-i)}0^\infty + {}^\infty 0.0^{2k+2}10^\infty = {}^\infty 0^\infty.$$

Thus, $\pi\delta^T$ is not preinjective and dually $\pi\delta$ is not surjective. On the other hand, suppose $\delta \in X$ is not surjective. Again taking the dual, we have $\pi\delta^T(x) = {}^\infty 0^\infty$ for some 0-finite x. If i is the leftmost nonzero coordinate of x, then $\delta_i = 1$ (as otherwise $\pi\delta^T(x)_{i-1} = 1$). We must then have $\delta_{i+1} = 0$ and $x_{i+1} = 1$ to account for the bit set by $\delta_i = 1$, $x_i = 1$. Since $\delta \in X$, we have $\delta_{i+2} = 0$, so that $x_{i+2} = 0$. Inductively, we see that for all $k \in \mathbb{N}$, $x_{i+k} = 1$ if and only if k is odd, and $\delta_{i+k} = 0$ for odd k. Thus, x is not 0-finite, so $\pi\delta^T$ must be preinjective and $\pi\delta$ surjective.

Proposition 7. *Let $\pi 0$ and $\pi 1$ be linear local rules on $\mathbb{Z}_p$ realizing $X \subset \{0,1\}^{\mathbb{Z}}$.*

- *If the left and right radii of $\pi 0$ are strictly larger than those of $\pi 1$, then*

$$\exists m : \forall u, v \in \{0,1\}^* : uv \sqsubset X \iff u0^m v \in X.$$

- *If $\pi 1$ has smaller left radius but larger right radius, then*

$$\exists k, m : \forall u, v \in \{0,1\}^* : u0^k v \sqsubset X \iff u0^{k+m} v \in X.$$

Proof. We only consider the first case, the second being similar.

First, we take the minimal common neighborhood for $\pi 0$ and $\pi 1$. By composing the rules with a power of the shift, we may assume this neighborhood is $[0, k-1]$. The local rules are defined by a vector of coefficients. Let $V^0, V^1 \in (\mathbb{Z}_p)^k$ be the vectors of coefficients for $\pi 0$ and $\pi 1$, respectively, so that

$$\pi i(g_0, \ldots, g_{k-1}) = V_0^i * g_0 + \cdots + V_{k-1}^i * g_{k-1}$$

Because the radii of $\pi 0$ (with respect to the minimal neighborhood) are strictly larger than those of $\pi 1$, we have $V_0^1 = V_{k-1}^1 = 0$ and $V_0^0, V_{k-1}^0 \neq 0$. The set of forbidden words of X are the words $w \in \{0,1\}^*$ such that the map $(\pi w)^T : \mathbb{Z}_p^{|w|} \to \mathbb{Z}_p^{|w|+k-1}$ is not preinjective.

For a vector $\vec{v} \in \mathbb{Z}_p^k$, define $\sigma(\vec{v})_{k-1} = 0$ and $\sigma(\vec{v})_i = \vec{v}_{i+1}$. We define a function f on $0 \cdot \mathbb{Z}_p^{k-1}$ (a subset of $\mathbb{Z}_p^k$) by $f(\vec{v}) = \sigma(\vec{v}) + a(\vec{v}) * V^0$, where $a(\vec{v}) \in \mathbb{Z}_p$ is such that $(\sigma(\vec{v}) + a(\vec{v})V^0)_0 = 0$. The function f is well-defined since $V_0^0 \neq 0$ and reversible since $V_{k-1}^0 \neq 0$. By reversibility, there exists a smallest positive integer m such that $f^m = \mathrm{id}$. We claim that 0^m can be inserted or removed anywhere in any point of X.

In the rest of the proof, when $w \in \{0,1\}^*$, we write $(\pi w)^T : (\mathbb{Z}_p)^{|w|} \to (\mathbb{Z}_p)^{|w|+k-1}$ for the map

$$(\pi w)^T(\vec{v}) = \sum_{i=0}^{|w|-1} 0^i \cdot \vec{v}_i V^{w_i} \cdot 0^{|w|-i-1},$$

which represents the dual of the map $\pi w : (\mathbb{Z}_p)^{|w|+k-1} \to (\mathbb{Z}_p)^{|w|}$ (in the notation of Lemma 3). As in the proof of Lemma 11, we see that it is enough to show that $(\pi uv)^T$ is injective if and only if $(\pi u 0^m v)^T$ is.

First, suppose $(\pi uv)^T$ is not injective. If already $(\pi u)^T$ or $(\pi v)^T$ is not injective, $(\pi u 0^m v)^T$ cannot be injective either. Otherwise, suppose that $(\pi uv)^T(ww') = 0$, where $|w| = |u|$ and $|w'| = |v|$. Clearly, we have $(\pi u)^T(w) = 0^{|u|-1}\vec{v}$ for some $\vec{v} \in 0\mathbb{Z}_p^{k-1}$. Denoting $A = a(\vec{v}) \cdot a(f(\vec{v})) \cdots a(f^{m-1}(\vec{v}))$, we have

$$(\pi u 0^m v)^T(w \cdot A \cdot w') = 0^{|uv|+m+k-1},$$

since $(\pi u 0^m)^T(w \cdot A) = 0^{|u|+m-1}\vec{v}$.

For the other direction, suppose

$$(\pi u 0^m v)^T(w \cdot A \cdot w') = 0^{|uv|+m+k-1}$$

for some w, A, w'. Again, $(\pi u)^T(w) = 0^{|u|-1}\vec{v}$. By induction, it is easy to see that, again,

$$A = a(\vec{v}) \cdot a(f(\vec{v})) \cdots a(f^{m-1}(\vec{v})),$$

since the A_i must be chosen so that $(\pi u)^T(wA_0 \cdots A_i)_{|w|+i} = 0$, and this is exactly what determines $a(f^i(\vec{v}))$. By the choice of m, we then have $(\pi u 0^m)^T(wA) = 0^{|u|+m-1}\vec{v}$, so that $(\pi uv)^T(ww') = 0^{|uv|+k-1}$. □

Example 2. *We take $v = 010$ and $u = 101$ as the coefficient vectors of $\pi 1$ and $\pi 0$, respectively. This corresponds to the realization of the even shift in Example1. It is easy to check that $f^2 = 0$, and as an even sequence of 0s stays even when one inserts the word 00 in it, the even shift has the expected property.*

Proposition 8. *The subshifts X_1 and X_3 in Question 2 are not in $\mathcal{H}_G$ when G is an abelian group of square-free order.*

Proof. First, consider rules $\pi 0$ and $\pi 1$ on $\mathbb{Z}_p$ for prime p, and suppose they realize a supershift X of $\mathcal{B}^{-1}(0^*1^*0^*)$. If $\pi 0$ and $\pi 1$ share a common extremal coordinate, then they realize the full shift. If $\pi 0$ does not have strictly larger radius in one of the directions, then simply by the bipermutivity of $\pi 1$, it is easy to see that $\mathcal{B}^{-1}(1^*0^*1^*) \subset X$. Finally, if $\pi 0$ has strictly larger left and right radii than $\pi 1$, then let m be such that 0^m can be inserted anywhere in any point of X, given by Proposition 7. Clearly, we have $\mathcal{B}^{-1}(1^*(0^m)^*1^*) \subset X$.

Now, suppose X_3 is in $\mathcal{H}_G$ by a realization π, and let $G = \mathbb{Z}_n$ for $n = \prod_i p_i$ for distinct primes p_i. It is easy to see that the local rules πi decompose into a cartesian product of local rules on the groups $\mathbb{Z}_{p_i}$ (this is where the square-freeness comes in). By the proof of Lemma 1, X_3 is then an intersection of subshifts in $\mathcal{L}_G$ for $G = \mathbb{Z}_p$ with p prime. By the previous paragraph, this is impossible: each of the subshifts also contains $\mathcal{B}^{-1}(1^*(0^{m_i})^*1^*)$ for some m_i, so that the intersection contains the subshift $\mathcal{B}^{-1}(1^*(0^{\prod_i m_i})^*1^*)$.

Similarly, we see that X_1 is not in $\mathcal{H}_G$. Namely, the proof of Lemma 9 shows that if a subshift realized by linear rules over $\mathbb{Z}_p$ contains $\mathcal{B}^{-1}(0^*1^* + 1^*0^*)$, then it also contains $\mathcal{B}^{-1}(0^*1^*0^*)$ or $\mathcal{B}^{-1}(1^*0^*1^*)$. By the previous paragraphs, it then contains the subshift $\mathcal{B}^{-1}(1^*(0^m)^*1^* + 0^*(1^m)^*0^*)$ for some m, and the intersection of such subshifts is not contained in X_1.

□

6 Other Properties

We briefly discuss realizability problems for some other dynamical properties. That is, again, to a finite set of local rules, we associate the set of nonuniform CA over them which give dynamics of the type we are interested in, when iterated, and ask the realization question of which subshifts (or other sets) occur as the set of such nonuniform CA. We give the combinatorial characterizations of the dynamical properties as the definitions. The corresponding general notions can be found in any standard reference, such as [9]. We begin with chain transitivity.

Definition 6. *Let Γ be a finite set of local rules over Σ and let $\gamma \in \Gamma^{\mathbb{Z}}$ be a nonuniform CA. If $r \in \mathbb{N}$ and $u, v \in \Sigma^{[-r,r]}$, a sequence of points*

$$x^0, y^0, x^1, y^1, \dots, x^n, y^n \in \Sigma^{\mathbb{Z}}$$

such that $\pi\delta(x^i) = y^i$ and $y^i_{[-r,r]} = x^{i+1}_{[-r,r]}$ for all i, and $x^0_{[-r,r]} = u, y^n_{[-r,r]} = v$ is called a chain *from u to v. If for $r \in \mathbb{N}$, a chain exists between all pairs $u, v \in \Sigma^{[-r,r]}$, then we say γ is* r-chain transitive*. If γ is r-chain transitive for all r, then we say it is* chain transitive*.*

Proposition 9. *For any finite set of local rules Γ, the chain transitive elements of $\Gamma^{\mathbb{Z}}$ form a subshift.*

Proof. It is clear that the set of chain transitive sequences is closed under the shift. We thus only need to show that it is closed. We only need to show, for a fixed r, that the set of r-chain transitive sequences is a closed set, as the chain transitive sequences are exactly their intersection.

So, suppose that γ is r-chain transitive and let $u, v \in \Sigma^{[-r,r]}$ be arbitrary. Let $x^0, y^0, x^1, y^1, \dots, x^n, y^n$ be a chain between u and v, and let $\gamma'_{[-r,r]} = \gamma_{[-r,r]}$. Then clearly $x^0, z^0, x^1, z^1, \dots, x^n, z^n$ where $z^i = \gamma'(x^i)$ is a chain between u and v for γ', so that γ' is r-chain transitive. ◻

A similar, but better-known property is transitivity.

Definition 7. *Let Γ be a finite set of local rules over Σ and let $\gamma \in \Gamma^{\mathbb{Z}}$ be a nonuniform CA. We say γ is* transitive *if*

$$\forall r \in \mathbb{N} : \forall x, y \in \Sigma^{\mathbb{Z}} : \exists n \in \mathbb{N} : \gamma^n(x)_{[-r,r]} = y_{[-r,r]}.$$

We do not know examples where the set of transitive sequences over a finite set of local rules is not a subshift.

Question 5. *Is there a set of local rules Γ such that the nonuniform CA in $\Gamma^{\mathbb{Z}}$ with transitive dynamics do not form a subshift?*

Lemma 12. *All SFTs can be realized through chain transitivity and transitivity.*

Proof. We take the rules given by Theorem 1 and compose them with a large power of the shift as in Lemma 5. The sequences that were nonsurjective before cannot become surjective after the composition, so they in particular do not become chain transitive or transitive. On the other hand, it is easy to see that the sequences that were surjective before the composition become transitive and chain transitive for any large enough power of the shift. □

In [2], it is proved that the only SFTs are realized through number conservation, where number conservation means that the alphabet Σ is a subset of $\mathbb{N}$, and the sum of the repeated pattern of a periodic point is preserved in the application of the automaton. Number conservation is a much studied notion in the theory of cellular automata, see for example [10, 11, 12]. Thus, it is interesting to ask realizability questions also for this class: which SFTs can be realized through number conservation? In fact, the proof in [2] already answers this.

Proposition 10. *Subshifts realized through number conservation are preimages of finite periodic SFTs in symbol projections.*

Proof. We show that if π is injective, then X is a finite periodic SFT. From this, the result follows trivially. By the results of [2], X is an SFT. In fact, they prove that for local rules given by $\pi : \Delta \to \Gamma$ where πd have left and right radius r, the forbidden patterns are words $\psi \in \Delta^{2r+1}$ such that

$$\exists u \in \Sigma^{2r+1} : \pi(\psi_{2r})(u) \neq u_0 + \sum_{i=0}^{2r-1} \pi(\psi_{i+1})(0^{2r-i}u_{[1,i+1]}) - \pi(\psi_i)(0^{2r-i}u_{[0,i]}).$$

Thus, if $\delta \in X$, the word $\delta_{[j-2r,j-1]}$ uniquely determines δ_j for all j. Clearly, X must then be a finite periodic SFT. □

We skip any further analysis of realizability of SFTs through number convervation.

Finally, we mention some dynamical properties which do not always yield subshifts, showing that investigations into their realizability would need to be made in more general frameworks. We only give the examples, as checking the properties is very easy. Of course, results such as Theorem 2 might hold for some of these classes. Equicontinuity and sensitivity were already considered in [2], and it was proved that for a subclass of linear nonuniform cellular automata, the set of sequences belonging to one of these classes is recognized by a Büchi automaton.

The definition of equicontinuity is the standard one: $\gamma \in \Gamma^{\mathbb{Z}}$ is said to be equicontinuous if the family $\{\gamma^i : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}} \mid i \in \mathbb{N}\}$ of functions is equicontinuous in the sense of Definition 10 given in the next section. The definition of sensitivity can be found in any standard reference. Nilpotency and asymptotic nilpotency are less standard notions, and we define them below.

Definition 8. *Let Γ be a finite set of local rules over Σ and let $\gamma \in \Gamma^{\mathbb{Z}}$ be a nonuniform CA. We say γ is* nilpotent *if*

$$\exists n : \forall x \in \Sigma^{\mathbb{Z}} : \gamma^n(x) = {}^{\infty}0^{\infty}.$$

We say γ *is* asymptotically nilpotent *if*

$$\forall x \in \Sigma^{\mathbb{Z}} : \forall i \in \mathbb{Z} : \exists n \in \mathbb{N} : \forall m \geq n : \gamma^m(x)_i = 0.$$

Lemma 13. *Each of the properties (non-)nilpotency, (non-)asymptotic nilpotency, (non-)equicontinuity, (non-)sensitivity, nonsurjectivity, (non-)injectivity, non-number conservation, non-chain transitivity and nontransitivity realizes a set that is not a subshift.*

Proof. Nilpotency and asymptotic nilpotency: Let F be the elementary CA 128 (0s spread), and G the elementary CA 0 ($x \mapsto {}^{\infty}0^{\infty}$). Now, asymptotically nilpotent sequences $\gamma \in \{F, G\}^{\mathbb{Z}}$ are those sequences where $\gamma_i = G$ for some i. Nilpotent sequences are those where G appears with bounded gaps.

Non-nilpotency and non-asymptotic nilpotency: Let F be the elementary CA 204 (the identity map) and G the elementary CA 0. The non-nilpotents and non-asymptotic nilpotents are all sequences except the cellular automaton with local rule G. This set is not closed.

(Non-)equicontinuity, (non-)sensitivity, nontransitivity: Let F be the elementary CA 150 (three-neighbor xor) and G the elementary CA 0. Now equicontinuous maps are those with infinitely many G in each direction, and this set of sequences is not closed or open. In this case, sensitive maps are exactly the complement, and thus not closed or open either.

In general, the complement of a property that can induce a nontrivial subshift does not always induce a subshift because the complement of a nonempty subshift within a larger subshift is never a subshift. Thus, nonsurjectivity, noninjectivity, non-number conservation, non-chain transitivity and non-transitivity do not always yield subshifts. □

7 Generalizations and Some Symbolic Dynamical Context

We briefly look into nonuniform CA defined over a subshift and other generalizations, from a symbolic dynamics perspective. We give a topological explanation for why $\mathcal{S}$ contains only subshifts (by way of proving such a result in high generality). We also discuss the possibility of considering nonuniform CA on general sofic shifts, and show, for example, that the set of nonuniform CA that are well-defined on a given sofic shift form a sofic shift themselves. We also give a dynamical characterization of nonuniform CA over finitely many local rules, and give a type of uniform invertibility condition for invertible nonuniform CA.

In the following, we prove sets to be subshifts (closed) in detail, but proofs of soficity of subshifts are omitted as standard, and are always 'true by definition' using the following observations:

- Any subshift is defined by its language, regular languages corresponding to the sofic shifts, and regular languages are characterized by monadic second-order logic on words [13].

- Any language defined by first-order logic with variables ranging over regular languages can be thought of as a monadic second-order statement about positions (coding words as subsets).

Definition 9. *Let $f : X \to Y$ be a continuous surjection, and $g : X \to X$ a continuous function. We define*

$$\mathrm{SF}(f, g) = \{y \in Y \mid \forall x \in f^{-1}(y) : \exists x' \in X : x = g(x')\}.$$

Lemma 14. *If X is compact and metrizable, $f : X \to Y$ is a continuous and open surjection and $g : X \to X$ is continuous, then $\mathrm{SF}(f, g)$ is closed.*

Proof. Suppose $y \notin \mathrm{SF}(f, g)$, so that for some $x \in X$, we have $f(x) = y$ and $x \notin g(X)$. Since X is compact, $g(X)$ is compact and thus closed, so $X \setminus g(X)$ is an open neighborhood of x. Since f is open, $f(X \setminus g(X))$ is an open neighborhood of y. Suppose $y' \in f(X \setminus g(X))$, and choose $x' \in X \setminus g(X)$ such that $f(x') = y'$. Since x' does not have a g-preimage, we have $y' \notin \mathrm{SF}(f, g)$. Thus, $y \in f(X \setminus g(X)) \subset Y \setminus \mathrm{SF}(f, g)$, and $\mathrm{SF}(f, g)$ is closed. □

Proposition 11. *If X and Y are subshifts, $f : X \to Y$ is an open factor map, and $g : X \to X$ is a cellular automaton, then $\mathrm{SF}(f, g)$ is a subshift of Y. If X is sofic, then so is $\mathrm{SF}(f, g)$.*

Proof. First, $\mathrm{SF}(f, g)$ is trivially shift-invariant, and it is closed by Lemma 14. Now, suppose X is sofic, so that Y is sofic as well. Then $\mathrm{SF}(f, g)$ is defined by a first-order statement about points of X and Y and cellular automata, and is thus sofic. □

Let Δ be a finite alphabet, and π a projection onto local rules Γ. Let $X = \Delta^{\mathbb{Z}} \times \Sigma^{\mathbb{Z}}$, f the projection onto the $\Delta^{\mathbb{Z}}$ component, and g the map $g(\delta, y) = (\delta, \pi\delta(y))$. Then, $\mathrm{SF}(f, g)$ is the set of sequences of local rules such that the corresponding nonuniform CA is surjective. Since f is a projection, it is open, so $\mathrm{SF}(f, g)$ is sofic by Proposition 11. Thus, Proposition 11 is a generalization of Theorem 10 of [2]. Note that in the case of nonuniform cellular automata, the sequence δ is unchanged under the action of g, that is, $f = f \circ g$. This discussion shows that while nonuniform CA are generalizations of cellular automata, families of nonuniform cellular automata can be seen as special cases of cellular automata on a larger alphabet. This is also essentially the content of Theorem 3.3 in [1].

Next, we show that for any set of local rules, the set of nonuniform CA over those rules that are well-defined on a sofic shift $Y \subset \Sigma^{\mathbb{Z}}$ (that is, map Y to itself) themselves form a sofic shift. We again start with a more general lemma.

Lemma 15. *Let X be compact and metrizable, Z a closed subset of X, $\pi_X : Y \times X \to X$ and $\pi_Y : Y \times X \to Y$ the natural projections, and $g : Y \times X \to Y \times X$ be continuous, then the set*

$$\mathrm{NF}(g, Z) = \{y \in Y \mid \forall x \in Z : \pi_X(g((y, x))) \in Z\}$$

is closed.

Proof. Suppose $y \notin \mathrm{NF}(g, Z)$, so that for some $x \in Z$, $\pi_X(g((y,x))) \notin Z$. Since $X \setminus Z$ is an open neighborhood of $\pi_X(g((y,x)))$ and $\pi_X \circ g$ is continuous, there exists an open neighborhood $U \ni (y,x)$ such that $\pi_X(g(U)) \subset X \setminus Z$. Then $\pi_Y(U)$ is an open neighborhood of y contained in $X \setminus \mathrm{NF}(g, Z)$. □

Proposition 12. *Let Δ be an alphabet, let $\pi : \Delta \to \Gamma$ be a projection onto a set of local rules, and let $Z \subset \Sigma^{\mathbb{Z}}$ be a subshift. Let $W \subset \Delta^{\mathbb{Z}}$ be the set of sequences δ such that $\pi\delta(Z) \subset Z$. Then W is a subshift. If Z is sofic, then so is W.*

Proof. It is clear that W is shift-invariant since Z is, and it is closed by Lemma 15, setting $X = \Sigma^{\mathbb{Z}}$, $Z = Z$, $Y = \Delta^{\mathbb{Z}}$ and $g(\delta, x) = (\delta, \pi\delta(x))$, as $W = \mathrm{NF}(g, Z)$. If Z is sofic, the soficity of W follows as in Proposition 11. □

Combining Proposition 12 and Proposition 11, we also get the following:

Proposition 13. *For a sofic shift $Z \subset \Sigma^{\mathbb{Z}}$ and a projection $\pi : \Delta \to \Gamma$ onto local rules Γ, the set of sequences $\delta \in \Delta^{\mathbb{Z}}$ such that $\pi\delta(Z) = Z$ is a sofic shift.*

Proof. The set W of sequences δ such that $\pi\delta(Z) \subset Z$ is a sofic subshift by Proposition 12. As in the discussion above, we define $X = W \times Z$, $g : X \to X$ by $g((\delta, z)) = (\delta, \pi\delta(z))$, $Y = W$, $f : X \to Y$ by $f((\delta, z)) = \delta$. Then $\mathrm{SF}(f, g)$ is sofic by Proposition 11, and is clearly exactly the set of sequences $\delta \in \Delta^{\mathbb{Z}}$ such that $\pi\delta(Z) = Z$. □

These propositions show that it makes sense to ask realizability questions also for nonuniform cellular automata over a sofic shift Z. For a sofic shift $Z \subset \Sigma^{\mathbb{Z}}$, one can ask, for example, which sofic shifts can be realized as the set of well-defined nonuniform CA on Z (Proposition 12), which as surjective well-defined nonuniform CA on Z (Proposition 13), and which as the surjective CA with the constraint that all sequences of local rules must be well-defined on Z.

Since CA are characterized as the shift-commuting continuous functions and nonuniform CA as the continuous functions, it also makes sense to ask whether nonuniform CA with finitely many local rules have such a characterization, or whether the definition is inherently combinatorial. Nonuniform CA with finitely many different local rules are called $r\nu$-CA in [1]. We supply a dynamical characterization, which suggests that such nonuniform CA are, in some sense, characterized as the *spatially equicontinuous maps.* For this, we give the general definition of equicontinuity of a family of maps.

Definition 10. *Let $f_i : X \to Y$ for $i \in I$ be a family of functions, where X and Y are metric spaces. We say the family $\{f_i \mid i \in I\}$ is* equicontinuous *if*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x, y \in X : d_X(x, y) < \delta \implies \forall i : d_Y(f_i(x), f_i(y)) < \epsilon$$

Proposition 14. *Let X be a subshift, and let $f : X \to X$. Then the following are equivalent*

- *the family $f_i = \sigma^{-i} \circ f \circ \sigma^i$ for $i \in \mathbb{Z}$ is equicontinuous on X.*

- *f is a nonuniform CA over finitely many local rules.*

Proof. Suppose $f = \gamma \in \Gamma^{\mathbb{Z}}$ where Γ is a finite set of local rules. Let r be a common radius for the local rules Γ. If $x_{[-r,r]} = y_{[-r,r]}$, then

$$f_i(x)_0 = \sigma^{-i}(\pi\delta(\sigma^i(x)))_0 = \pi\delta(\sigma^i(x))_{-i} = \pi(\delta_{-i})(x_{[-r,r]}) = \cdots = f_i(y)_0.$$

Conversely, if f_i is equicontinuous, then $f = f_0$ is in particular continuous, and thus $f = \gamma$ for some $\gamma \in \Gamma^{\mathbb{Z}}$ where Γ is a possibly infinite set of local rules. Since the family of f_i is equicontinuous, there exists r such that for any $i \in \mathbb{Z}$ and $x, y \in X$ such that $x_{[-r,r]} = y_{[-r,r]}$, we have $f_i(x)_0 = f_i(y)_0$. Clearly this means that we can choose the radius r for each local rule γ_i, so Γ can be chosen finite. □

We end this article with a simple general result extending Theorem 2: the case where the injective sequences form an SFT is exactly the one where we can invert the action of any CA uniformly, by a block map.

Definition 11. *Let Δ be a finite alphabet, and $\pi : \Delta \to \Gamma$ a projection onto a finite set of local rules over Σ. We say (π, Σ) is* uniformly invertible *if there exists a finite set of local rules Θ on Σ and a block map $\psi : \Delta^{\mathbb{Z}} \to \Theta^{\mathbb{Z}}$ (where $\Theta^{\mathbb{Z}}$ has the usual product topology) such that whenever $\pi\delta \in \Gamma^{\mathbb{Z}}$ is injective for $\delta \in \Delta^{\mathbb{Z}}$, we have*

$$\psi(\delta) \circ \pi\delta = id_{\Sigma^{\mathbb{Z}}}.$$

Note that we do not require $\gamma \circ \psi(\gamma) = \mathrm{id}_{\Sigma^{\mathbb{Z}}}$, and we require nothing from $\psi(\delta)$ (except its existence) when $\pi\delta$ is not injective.

Theorem 3. *Let Δ be a finite alphabet, and $\pi : \Delta \to \Gamma$ a projection onto a finite set of local rules over Σ. Let*

$$X = \{\delta \in \Delta^{\mathbb{Z}} \mid \pi\delta : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}} \text{ is injective}\}.$$

Then the following are equivalent.

- *X is closed*
- *X is an SFT.*
- *(π, Σ) is uniformly invertible.*

Proof. The equivalence of the first two conditions is the content of Theorem 2. Now, suppose X is an SFT. Let $Y = X \times \Sigma^{\mathbb{Z}}$, and let $f : Y \to Y$ be the CA $f((\delta, x)) = (\delta, \pi\delta(x))$. By the assumption, f is injective, and thus bijective onto its image Z, which is automatically a subshift. The block map $f : Y \to Z$ has a continuous inverse $g : Z \to Y$, which is also necessarily a block map. Note that it is possible that Z is a proper subshift of Y.

Now, observe that since f behaves as the identity map on the left component, g does as well, and $\pi_1 Z = X$, where π_1 is the projection to the first component X

of Y. Let $G : A \to \Delta \times \Sigma$ be the local rule of g of radius r, where $A = B \times \Sigma^{[-r,r]}$, and $B \subset \Delta^{[-r,r]}$ is the set of words of length $2r + 1$ that occur in X.

We can extend g to a CA $h : Y \to Y$ as follows:

$$h((\delta, x))_i = \begin{cases} G(\delta_{[i-r,i+r]}, x_{[i-r,i+r]}), & \text{if } (\delta_{[i-r,i+r]}, x_{[i-r,i+r]}) \in A, \\ (\delta_i, x_i), & \text{otherwise.} \end{cases}$$

It is easy to see that indeed $h(Y) \subset Y$. Since $h|_Z = g$, we have $h \circ f = \mathrm{id}_Y$. Let H be the local rule of h, also of radius r.

Let Θ be the set of local rules over Σ of radius r. We define the block map $\psi : X \to \Theta^{\mathbb{Z}}$ as follows (where note that each $\psi(\delta)_i$ is an element of Θ):

$$\psi(\delta)_i(a_{-r}, \dots, a_r) = \pi_2(H(\delta_{[i-r,i+r]}, (a_{-r}, \dots, a_r))),$$

where π_2 is the projection from Y to the second component $\Sigma^{\mathbb{Z}}$ of Y. By the properties of h, for all $\delta \in X$ we have $\psi(\delta) \circ \pi\delta = \mathrm{id}_{\Sigma^{\mathbb{Z}}}$. Now, simply fill the local rule of ψ arbitrarily: it is impossible for $\psi(\delta) \circ \pi\delta = \mathrm{id}_{\Sigma^{\mathbb{Z}}}$ to hold for any additional points δ, since we assumed $\pi\delta$ is not injective for $\delta \notin X$.

Conversely, suppose (π, Σ) is uniformly invertible, and let ψ and Θ be as in Definition 11. Then

$$X = \{\delta \in \Delta^{\mathbb{Z}} \mid \forall i \in \mathbb{Z} : \forall x \in \Sigma : (\psi(\gamma) \circ \gamma)(x)_i = x_i\},$$

which is an SFT. □

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