On Stable and Unstable Limit Sets of Finite Families of Cellular Automata*

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Abstract. In this paper, we define the notion of limit set for a finite family of cellular automata, which is a generalization of the limit set of a single automaton. We prove that the hierarchy formed by increasing the number of automata in the defining set is infinite, and study the boolean closure properties of different classes of limit sets.

Keywords: cellular automata, symbolic dynamics, limit sets

1 Introduction

Cellular automata are discrete dynamical systems that, despite their simple definition, can have very complex dynamics. They are defined as transformations on a space of configurations, that is, infinite sequences of symbols, that operate by applying the same local rule in every coordinate. The research of CA dates back to the 60's [2].

We are interested in the long-term time evolution of the configuration space under the dynamics given by cellular automata, which can be studied using the concept of limit sets. The limit set of a cellular automaton consists of those configurations that can appear arbitrarily late in the evolution of the system. An automaton is called stable if its evolution actually reaches the limit set at some point in time. In general, limit sets can be very complicated, and it has been shown that all their nontrivial properties are undecidable, given the defining CA [4]. In this paper, we generalize this notion and define limit sets of finite families of cellular automata, as opposed to a single automaton.

The paper is organized as follows. Section 1 consists of this introduction. In Section 2 we define the notions used.

In Section 3, we prove some basic lemmas about limit sets of families of cellular automata. Using the notion of projective subdynamics and results from [7], we obtain a necessary condition for a subshift to be the limit set of some family of CA.

In Section 4, we focus on the classes of limit sets of a given number of cellular automata, and the hierarchy formed by increasing this number. The main results

* Research supported by the Academy of Finland Grant 131558

of this chapter state that both the stable and unstable hierarchies are infinite, with proper inclusions at every step. We also prove that the stable hierarchy, when restricted to transitive subshifts, collapses to a single level.

In Section 5, we study the closure properties of classes of limit sets under set-theoretic operations, that is, unions and intersections. The class of stable limit sets turns out to be closed under union, but our counterexamples show that neither the stable nor unstable class is closed under (nonempty) intersection. We also prove that the hierarchy formed by considering finite unions of stable limit sets of a given number of CA is finite: all stable limit sets of finite families of cellular automata can be expressed as finite unions of limit sets of just two automata. An open question is whether the hierarchy has only one level.

2 Definitions

Let S be a finite set, the *alphabet*, which is given the discrete topology. We denote by $S^* = \bigcup_{n \in \mathbb{N}} S^n$ the set of *words* over S, and if $w \in S^n$, we denote $|w| = n$. The space $S^{\mathbb{Z}}$ with the induced product topology is called the *full shift over* S. The topology of $S^{\mathbb{Z}}$ is also given by the metric d defined by

$$d(x, y) = 2^{-\min\{|i| \mid x_i \neq y_i\}}.$$

For a subset $X \subset S^{\mathbb{Z}}$ and $\epsilon > 0$, we define

$$B_\epsilon(X) = \{y \in S^{\mathbb{Z}} \mid \exists x \in X : d(x, y) < \epsilon\}.$$

We also consider the two-dimensional shift space $S^{\mathbb{Z}^2}$ with the product topology.

If $x \in S^{\mathbb{Z}}$, we denote the ith coordinate of x with x_i, and abbreviate the expression $x_i x_{i+1} \cdots x_{i+n-1}$ by $x_{[i,i+n-1]}$. If $u, v, w \in S^*$, we denote by ${}^\infty uv.wt^\infty$ the element $x \in S^{\mathbb{Z}}$ defined by $x_{[-n|u|,-(n-1)|u|-1]-|v|} = u$, $x_{[-|v|,-1]} = v$, $x_{[0,|w|-1]} = w$ and $x_{|w|+[n|t|,(n+1)|t|-1]} = w$, and the element ${}^\infty vv.vv^\infty$ is denoted ${}^\infty v^\infty$. We may use the notation $x = {}^\infty uvw^\infty$ when the position of the origin is irrelevant or can be inferred from the context. If $a \in S$, an element of the form ${}^\infty awa^\infty$ is called *a-finite*, and an element of the form ${}^\infty a^\infty$ is called a *uniform configuration*. For a word $w \in S^n$, we say that *w occurs in x* and denote $w \sqsubset x$, if there exists i such that $w = x_{[i,i+n-1]}$. On $S^{\mathbb{Z}}$ we define the *shift map* σ_S (or simply σ, if S is clear from the context) by $\sigma_S(x)_i = x_{i+1}$ for all i. Clearly σ is bijective and continuous w.r.t. the topology of $S^{\mathbb{Z}}$.

A finite set of words $W \subset S^*$ is said to be *mutually unbordered* if whenever two words $v, w \in W$ occur in some $x \in S^{\mathbb{Z}}$, say $v = x_{[0,|v|-1]}$ and $w = x_{[m,m+|w|-1]}$ where $m \geq 0$, then we must have $m \geq |v|$. Note that we may have $v = w$ in this definition. Given any configuration $x \in S^{\mathbb{Z}}$, the set W now uniquely partitions $\mathbb{Z}$ into $|W| + 1$ disjoint sets A_w for $w \in W$ and A, defined by $i \in A_w$ if and only if $x_{[m,m+|w|-1]} = w$ for some $m \in [i - |w| + 1, i]$, and A being the complement of $\bigcup_{w \in W} A_w$.

A closed, shift-invariant subset of $S^{\mathbb{Z}}$ is called a *subshift*. Alternatively, given a set of *forbidden words* $F \subset S^*$, a subshift can be defined by those points of

$S^{\mathbb{Z}}$ in which no word from F occurs. If F is finite, the resulting subshift is a *subshift of finite type*, abbreviated SFT. We define $\mathbb{Z}^2$ subshifts and $\mathbb{Z}^2$ SFT's analogously, as closed shift-invariant subsets of $S^{\mathbb{Z}^2}$, which are also defined by a set of forbidden two-dimensional patterns. The set of words of length n occurring in a subshift X is denoted by $\mathcal{B}_n(X)$, and we define $\mathcal{B}(X) = \bigcup_n \mathcal{B}_n(X)$. If X has the property that for all $v, w \in \mathcal{B}(X)$ and $N \in \mathbb{N}$ there exists an $n \geq N$, a word $z \in S^n$ and a point $x \in X$ with $vzw \sqsubset x$, then we say that X is *transitive*. If there exists an N such that the previous holds for all $n \geq N$, we say that X is *mixing*. A point $x \in X$ is *doubly transitive*, if for all $w \in \mathcal{B}(X)$ and for all $N \in \mathbb{N}$, there exist $m \leq -N$ and $n \geq N$ with $x_{[m,m+|w|-1]} = x_{[n,n+|w|-1]} = w$. Clearly every transitive subshift contains a doubly transitive point. The restriction of σ_S to X is denoted by σ_X. The *entropy* of a subshift X is defined as $h(X) = \lim_{n \longrightarrow \infty} \frac{1}{n} \log |\mathcal{B}_n(X)|$.

Given an $n \times n$ integral matrix M with $M_{ij} \geq 0$ for all i and j, we can construct an SFT from it by taking $S = \{(i, j, m) \mid i, j \in [1, n], 0 \leq m < M_{ij}\}$ as the state set and $\{((i_1, j_1, m_1), (i_2, j_2, m_2)) \mid j_1 \neq i_2\}$ as the set of forbidden words. This SFT is called the *edge shift defined by* M. If there is an N such that $(M^n)_{ij} > 0$ for all i and j whenever $n \geq N$, we say that M is *primitive*. It is known that M is primitive if and only if its edge shift is mixing.

Let X and Y be two subshifts. A *block map from X to Y* is a continuous map $\psi : X \to Y$ such that $\psi \circ \sigma_X = \sigma_Y \circ \psi$. It is known that all block maps are defined by *local rules* $\Psi : \mathcal{B}_{2r+1}(X) \to \mathcal{B}_1(Y)$ so that $\psi(x)_i = \Psi(x_{[i-2r,i+2r]})$ for all $x \in X$ and $i \in \mathbb{N}$. The number r is called the *radius* of ψ. If $r = 1$ and Ψ does not depend on the rightmost coordinate, we say that ψ has radius $\frac{1}{2}$ and give Ψ as a function from $\mathcal{B}_2(X)$ to $\mathcal{B}_1(Y)$. The block map ψ is said to be *preinjective* if $\psi(x) \neq \psi(y)$ whenever $x_i = y_i$ for all but a finite number of i. If ψ is surjective, it is called a *factor map*, and then Y is a *factor* of X. A factor of an SFT is called *sofic*. A *cellular automaton* is a block map from $S^{\mathbb{Z}}$ to itself.

Let $\mathcal{F}$ be a finite family of cellular automata. We define the sets $L_i(\mathcal{F})$ for $i \in \mathbb{N}$ by $L_0(\mathcal{F}) = S^{\mathbb{Z}}$, and $L_i(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} f(L_{i-1}(\mathcal{F}))$ for $i > 0$. The *limit set of* $\mathcal{F}$ is the set $L(\mathcal{F}) = \bigcap_{i \in \mathbb{N}} L_i(\mathcal{F})$. We say that $\mathcal{F}$ is *stable* if $L(\mathcal{F})$ is equal to one of the $L_i(\mathcal{F})$.

We denote by $k-\mathrm{LIM}_s$ and $k-\mathrm{LIM}_u$ the classes of stable and unstable limit sets of families of at most k cellular automata, respectively. The notation $k-\mathrm{LIM}_x$ refers to both classes (not their union), thinking of x as a variable ranging over $\{s, u\}$. We also denote $\infty-\mathrm{LIM}_x = \bigcup_k k-\mathrm{LIM}_x$ for $x \in \{s, u\}$.

To illustrate the concept of limit sets of finite families of CA, we give an example of a complex limit set of two automata.

Example 1. Consider the two automata f_0 and f_1 on the alphabet $\{0, 1, \#\}$ where each f_i has radius $\frac{1}{2}$, and the local rule of f_i is given by the following table:

	0	1	#
0	0	0	#
1	1	1	#
#	i	i	#

Now the limit set $L(\{f_0, f_1\})$ is the subshift defined by the forbidden words $\{\#uv\#w \mid n \in \mathbb{N}, u, w \in \{0,1\}^n, v \in \{0,1,\#\}^*, u \neq w\}$.

3 Basic Results

The following two lemmas are direct generalizations of well-known properties of limit sets of a single cellular automaton.

Lemma 1. *If $\mathcal{F}$ is a finite family of cellular automata, then $L(\mathcal{F}) = \bigcup_{f \in \mathcal{F}} f(L(\mathcal{F}))$.*

Proof. It is clear that $f(L(\mathcal{F})) \subset L(\mathcal{F})$ for all $f \in \mathcal{F}$, since

$$f(L(\mathcal{F})) = f(\bigcap_i L_i(\mathcal{F})) \subset \bigcap_i f(L_i(\mathcal{F})) \subset \bigcap_i L_{i+1}(\mathcal{F}) = L(\mathcal{F}).$$

For the other inclusion, consider an arbitrary point $x \in L(\mathcal{F})$, and let $Z = \bigcup_{f \in \mathcal{F}} f^{-1}(x)$. Since $x \in L_{i+1}(\mathcal{F})$, we have $Z \cap L_i(\mathcal{F}) \neq \emptyset$ for all i. Since we now have a descending chain of nonempty compact sets, the intersection

$$\bigcap_i (Z \cap L_i(\mathcal{F})) = Z \cap L(\mathcal{F})$$

is nonempty. Therefore, x has a preimage in $L(\mathcal{F})$.

Lemma 2. *Let $\mathcal{F}$ be a finite family of cellular automata. For all $\epsilon > 0$, there exists k such that $L_k(\mathcal{F}) \subset B_\epsilon(L(\mathcal{F}))$.*

Proof. The set $Z = S^{\mathbb{Z}} - B_\epsilon(L(\mathcal{F}))$ is compact. Therefore, if $Z \cap L_i(\mathcal{F})$ were nonempty for all i, we would also have $Z \cap L(\mathcal{F}) \neq \emptyset$ as in the previous proof.

By the definition of the metric in $S^{\mathbb{Z}}$, this means that for all m, there exists a k_m such that $\mathcal{B}_m(L_k(\mathcal{F})) = \mathcal{B}_m(L(\mathcal{F}))$ for all $k \geq k_m$. In particular, a limit set that is an SFT must be stable.

We will use the following result from [7] to prove certain languages not to be unstable limit sets of any families of cellular automata. First, we define the notion of projective subdynamics.

Definition 1. *Let X be a $\mathbb{Z}^2$ SFT. The set of horizontal rows appearing in points of X is called the $\mathbb{Z}$-*projective subdynamics *of X.*

Definition 2. *If X is a one-dimensional sofic shift, we say l is a* universal period *for X if there exists M such that for all $x \in X$ there exists y with $y = \sigma^l(y)$ such that $|\{i \mid x_i \neq y_i\}| \leq M$.*

Lemma 3. ([7]) *A zero-entropy proper one-dimensional sofic shift X is realizable as the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ SFT if and only if it has no universal period.*

Corollary 1. *A zero-entropy proper sofic shift with a universal period is not the limit set of any finite family of CA.*

Proof. Any family $\mathcal{F}$ of cellular automata defined on $S^{\mathbb{Z}}$ defines a $\mathbb{Z}^2$ SFT X over the alphabet $S \times \mathcal{F}$ as follows: The $\mathcal{F}$-component must be constant in every horizontal row. If $f \in \mathcal{F}$ is the CA of row $i+1$, then the S-component of row i must be the f-image of the S-component of row $i+1$. Now the $\mathbb{Z}$-projective subdynamics of X is precisely $L(\mathcal{F}) \times \{{}^{\infty}f^{\infty} \mid f \in \mathcal{F}\}$, so it cannot be a zero-entropy proper sofic shift with a universal period. But then $L(\mathcal{F})$ cannot have a universal period either.

We present here a realization theorem for limit sets, which uses techniques already found in [7].

Theorem 1. *If X is the limit set of a family of cellular automata containing at least two periodic points, then X is realizable as the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ SFT.*

Proof. Let S be the alphabet of $X = L(\{f_1, \ldots, f_k\})$, and let $q_1, q_2 \in X$ be distinct periodic points such that $q_1 = {}^{\infty}a^{\infty}$ and q_2 has period p. Using the identification $S^{\mathbb{Z}^2} \cong (S^{\mathbb{Z}})^{\mathbb{Z}}$, we think of configurations of $S^{\mathbb{Z}^2}$ as bi-infinite vertical words over the alphabet $S^{\mathbb{Z}}$ of bi-infinite horizontal words. Then, choose k mutually unbordered vertical words $w_i \in \{q_1, q_2\}^m$.

We now construct the $\mathbb{Z}^2$ SFT Y having X as its $\mathbb{Z}$-projective subdynamics. The local rule of Y works on a coordinate (a, b) of a configuration x as follows: If $x_{[a,a+p-1]\times[b+r,b+r+m-1]}$ does not appear in any w_i for any $r \in [1-m, 0]$, then both $x_{[a,a+p-1]\times[b+1,b+m]}$ and $x_{[a,a+p-1]\times[b-m,b-1]}$ must do so. Then, since the w_i are mutually unbordered, each row of Y must be either part of some vertical word w_i, or between two such words. Additionally, if $\substack{x \\ w_i \\ y}$ appears as a vertical subword in Y, then we require that $y = f_i(x)$. It is now clear that X is the $\mathbb{Z}$-projective subdynamics of Y.

Corollary 2. *All stable limit sets X of finite families of cellular automata are realizable as the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ SFT.*

Proof. Since X is stable, it must be sofic. If X is an SFT, the claim is trivial. Now, if $|X| > 1$ and X is proper sofic but does not contain two distinct periodic points, then it contains a unique periodic point, which must then be unary. But then it is clear that X has zero entropy and a universal period, contradicting Corollary 1.

4 Hierarchies

We now turn to the relations between the classes $k-\mathrm{LIM}_x$. The following theorem is obvious since all classes $k-\mathrm{LIM}_s$ contain SFT's, and all classes $k'-\mathrm{LIM}_u$ contain subshifts that are not sofic.

Theorem 2. *The classes k−LIM$_s$ and k'−LIM$_u$ are incomparable for all $k, k' \geq 1$.*

It is also known that the classes are not completely disjoint [1].

It is slightly more complicated to prove that both hierarchies are proper. This will be our goal for the remainder of this section. We begin by finding arbitrarily large families of mixing SFT's X_i that have at least one uniform configuration with the property that X_i does not factor onto X_j for any $i \neq j$. We need some lemmas from [5].

Definition 3. *If A is a primitive integral matrix, let λ_A be its greatest eigenvalue with respect to absolute value, and $\mathrm{sp}^\times(A)$ the unordered list (or multiset) of its eigenvalues, called the* nonzero spectrum *of A. We use the notation $\langle \lambda_1, \ldots, \lambda_k \rangle$ for the unordered list containing the elements λ_i.*

Lemma 4. ([5]) *The entropy of the edge shift X defined by a primitive integral matrix A is $\log \lambda_A$.*

Lemma 5. ([5]) *If the edge shifts X and Y defined by two primitive integral matrices A and B, respectively, have the same entropy and X factors onto Y, then $\mathrm{sp}^\times(B) \subset \mathrm{sp}^\times(A)$.*

The previous lemma gives us a necessary condition for factoring between edge shifts of equal entropy, so it is enough to find, for each k, a family of k matrices M_i with the property that $\lambda_{M_i} = \lambda_{M_j}$ for all i and j, but $\mathrm{sp}^\times(M_i) \not\subset \mathrm{sp}^\times(M_j)$ whenever $i \neq j$. The following lemmas are the tools we need for this.

Definition 4. *Let $\Lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$ be an unordered list of complex numbers. Denoting $\Lambda^d = \langle \lambda_1^d, \ldots, \lambda_k^d \rangle$, let*

$$\mathrm{tr}(\Lambda) = \sum_{i=1}^{k} \lambda_i$$

be the trace *of Λ, and*

$$\mathrm{tr}_n(\Lambda) = \sum_{d|n} \mu(\frac{n}{d}) \mathrm{tr}(\Lambda^d)$$

the nth trace *of Λ, for all $n \in \mathbb{N}$, where $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function.*

Lemma 6. ([5]) *Let A be a primitive integral matrix and B an integral matrix such that*

- *$\lambda_B < \lambda_A$, and*
- *$\mathrm{tr}_n(\mathrm{sp}^\times(A)) + \mathrm{tr}_n(\mathrm{sp}^\times(B)) \geq 0$ for all $n \geq 1$.*

Then there is a primitive integral matrix C such that $\mathrm{sp}^\times(C) = \mathrm{sp}^\times(A) \cup \mathrm{sp}^\times(B)$.

Now let A be a matrix $[\lambda]$ with a single entry much greater than k, and let B_i be the matrices $B_i = [i]$ for $i \in [1, k]$. By taking a large enough λ, the assumptions of Lemma 6 are satisfied for the pairs A and B_i. Now, the matrices C_i, where C_i is given by the lemma for A and B_i, are primitive integral matrices with the same greatest eigenvalue, but incomparable nonzero spectra. Thus, their edge shifts have the same entropy but none of them factor onto another one by Lemma 5.

We still have one problem left: the edge shifts defined by the C_i might not have uniform configurations. This is fixed by taking a common power of the C_i: Since the matrices are primitive, there exists a p such that all of C_i^p have nonzero matrix trace, and thus define edge shifts with uniform configurations. The C_i^p are again primitive, and since $\mathrm{sp}^\times(C_i^p) = \mathrm{sp}^\times(C_i)^p$ and the eigenvalues are positive, we still have no factoring relations. Renaming the symbols of the shifts, we have proved the following lemma:

Lemma 7. *For all $k \in \mathbb{N}$, there exists a finite alphabet S_k, a symbol $a \in S_k$ and a set $\{X_1, \ldots, X_k\}$ of k mixing edge shifts over S_k such that whenever $i \neq j$, we have that X_i does not factor onto X_j, $X_i \cap X_j = {}^\infty a^\infty$ and $\mathcal{B}_1(X_i) \cap \mathcal{B}_1(X_j) = a$.*

We still need some further lemmas:

Lemma 8. ([6]) *Let $X \subset S^{\mathbb{Z}}$ be an SFT. Then X is the stable limit set of some cellular automaton if and only if X is mixing and contains a uniform configuration.*

Lemma 9. ([5, Corollary 4.4.9]) *If X is an SFT and Y a transitive sofic shift with $X \subset Y$ and $h(X) = h(Y)$, then $X = Y$.*

Lemma 10. ([5, Proposition 4.1.9]) *If X is a subshift and ψ any block map, then $h(\psi(X)) \leq X$.*

We are now ready to prove the main theorems of this chapter.

Theorem 3. *For all k, $(k-1)-\mathrm{LIM}_s \subsetneq k-\mathrm{LIM}_s$.*

Proof. Let X_i be given by Lemma 7, and for each i let f_i be the cellular automaton given by Lemma 8 having X_i as its stable limit set, operating on $\mathcal{B}_1(X_i)$ instead of S_k. We extend each f_i to the common alphabet S_k by having f_i consider every symbol of $S_k - \mathcal{B}_1(X_i)$ as a. Then clearly the system $\{f_1, \ldots, f_k\}$ has $X = \bigcup_i X_i$ as its stable limit set, and thus $X \in k-\mathrm{LIM}_s$.

Now consider a hypothetical system $\mathcal{F}$ with $|\mathcal{F}| < k$ such that $X = L(\mathcal{F}) = L_n(\mathcal{F})$. Let i be arbitrary, and consider a doubly transitive point $x \in X_i$. Since x must have a preimage in X, we see that for some $f \in \mathcal{F}$ we have $X_i \subset f(X_j)$. By Lemma 10, this means that $h(X_i) \leq h(f(X_j)) \leq h(X_j) = h(X_i)$, so by Lemma 9 we have $X_i = f(X_j)$. But by the property of the X_i, this is only possible if $i = j$, so that f maps X_i onto itself.

Since $|\mathcal{F}| < k$, there must exist $f \in \mathcal{F}$ and indices i and j such that f maps both X_i and X_j onto themselves. In this case, we must have $f({}^\infty a^\infty) = {}^\infty a^\infty$.

Let r be the radius of f. Let $b' \in \mathcal{B}(X_i) - a^*$ such that ${}^\infty a b' a^\infty \in X_i$. Now there exists $x' \in X_i$ with $f^n(x') = {}^\infty a b' a^\infty$. Let $x = {}^\infty a x'_{[-rn,|b|+rn]} a^\infty$. Then $f^n(x) = {}^\infty a b a^\infty$, where $b \in \mathcal{B}(X_i)$ with $b_{[m,m+|b'|-1]} = b'$ for some m. Similarly, we obtain $c \in \mathcal{B}(X_j) - a^*$ such that ${}^\infty a c a^\infty$ has an a-finite f^n-preimage in X_j. For all $N \in \mathbb{N}$, define $y_N = {}^\infty a b a^N c a^\infty \in S_k^{\mathbb{Z}}$. Now for large enough N, $f^{-n}(y_N) \neq \emptyset$, which is a contradiction, since $y_N \notin X$.

For the unstable case, yet another lemma is needed:

Lemma 11. ([5, Theorem 8.2.19]) *If $X \subset S^{\mathbb{Z}}$ is a mixing SFT and $\phi : X \to X$ is a factor map, then ϕ is preinjective.*

Theorem 4. *For all k, $(k-1)$–$\mathrm{LIM}_u \subsetneq k$–$\mathrm{LIM}_u$.*

Proof. Let X_i be given by Lemma 7, and for each i let f_i be the cellular automaton given by Lemma 8 having X_i as its stable limit set, operating on $\mathcal{B}_1(X_i)$ instead of S_k. We extend each f_i to the alphabet $S_k \cup \{\#\}$ by having f_i consider every symbol of $S_k - \mathcal{B}_1(X_i)$ as a, and making $\#$ a spreading state. Denote $X' = \bigcup_i X_i$. It is clear that the family $\{f_1, \ldots, f_k\}$ has $X = \overline{\{{}^\infty \# b \#^\infty \mid b \sqsubset X'\}}$ as its unstable limit set, implying $X \in k$–LIM_u.

Now consider again a hypothetical system $\mathcal{F}$ with $|\mathcal{F}| < k$ having X as its unstable limit set. It is easy to see that a doubly transitive point $x \in X_i$ must have a preimage in X without the symbol $\#$. In particular, we again see that some $f \in \mathcal{F}$ must map X_i onto itself, and find f, i and j such that f maps both X_i and X_j onto themselves. It is clear that $f({}^\infty a^\infty) = {}^\infty a^\infty$.

Now it is not a contradiction that configurations with symbols from both $\mathcal{B}_1(X_i) - \{a\}$ and $\mathcal{B}_1(X_j) - \{a\}$ have long chains of preimages, so we need a slightly more involved argument. Let r be the radius of f. By Lemma 2, each point in $S_k^{\mathbb{Z}}$ must locally approach X when f is applied to it repeatedly, so there exists an M such that $\mathcal{B}_{2r+1}(f^m(S_k^{\mathbb{Z}})) \subset \mathcal{B}_{2r+1}(X)$ for all $m \geq M$.

Let $y = {}^\infty(b a^{2r(M+1)} c a^{2r(M+1)})^\infty$ where $b, c \notin a^*$, ${}^\infty a b a^\infty \in X_i$ and ${}^\infty a c a^\infty \in X_j$. Since y is periodic with some period p, all of $f^n(y)$ are periodic with period p. We claim that in all points $f^n(y)$ for $n \in \mathbb{N}$, symbols from both $\mathcal{B}_1(X_i) - \{a\}$ and $B_1(X_j) - \{a\}$ appear: First, no $\#$ can appear in the images, since all $(2r+1)$-blocks that occur will be from $\mathcal{B}_{2r+1}(X')$. Since f is a surjection from X_i to itself and X_i is mixing, f is preinjective on X_i by Lemma 11, and similarly for X_j.

Now, when we apply f to a block of the form $a^{2r} w a^{2r}$ where $w \sqsubset X_i$ is not completely over a, the result must contain at least one symbol of $\mathcal{B}_1(X_i) - \{a\}$, and similarly for X_j. Since non-a symbols from both subshifts cannot appear in the same $(2r+1)$-block after M steps, we then inductively see that symbols from both shifts must appear in all of the points $f^n(y)$. Since the $f^n(y)$ are periodic with period p, they cannot locally approach X, and this contradiction proves the claim.

Note that the more involved argument of Theorem 4 can also be used in the stable case, implying that there exist subshifts

$$X \in k\text{–}\mathrm{LIM}_s - ((k-1)\text{–}\mathrm{LIM}_s \cup (k-1)\text{–}\mathrm{LIM}_u),$$

and

$$Y \in k\text{–LIM}_u - ((k-1)\text{–LIM}_s \cup (k-1)\text{–LIM}_u)$$

for all $k > 1$.

The following theorem shows that the existence of several mixing components is important in the stable case:

Theorem 5. *If $X \in \infty\text{–LIM}_s$ is transitive, then $X \in 1\text{–LIM}_s$.*

Proof. Let X be the stable limit set of the family $\mathcal{F}$ reached in one step. Then, since X contains a doubly transitive point, we find $f \in \mathcal{F}$ that maps X onto itself as in the previous proofs. But the limit set of f must be contained in X, so $X = L(\{f\})$, and f reaches X in one step.

In the unstable case, Theorem 5 might not hold as such, since the automaton f could be stable. The following, however, is true.

Theorem 6. *If $X \in \infty\text{–LIM}_u$ is transitive, then $X \in 1\text{–LIM}_s \cup 1\text{–LIM}_u$.*

Question 1. Is the unstable hierarchy proper when restricted to transitive subshifts?

5 Boolean Operations

In this section, we study the relation between the classes $\infty\text{–LIM}_x$ and set-theoretic operations. We begin with an easy lemma.

Lemma 12. *If $\mathcal{F}$ and $\mathcal{F}'$ are stable families of cellular automata, then $L(\mathcal{F}) \cup L(\mathcal{F}')$ is the limit set of a stable family $\mathcal{F}''$ of automata. If both limit sets are reached in one step, then $\mathcal{F}''$ can be taken to be $\mathcal{F} \cup \mathcal{F}'$.*

Proof. Let k be such that $L_k(\mathcal{F}) = L(\mathcal{F})$ and $L_k(\mathcal{F}') = L(\mathcal{F}')$. Then $L_1(\mathcal{F}'') = L(\mathcal{F}'') = L(\mathcal{F}) \cup L(\mathcal{F}')$, where $\mathcal{F}'' = \mathcal{F}^k \cup \mathcal{F}'^k$.

Corollary 3. *The class $\infty\text{–LIM}_s$ is closed under union.*

Theorem 7. *Let $\mathcal{F}$ be a finite family of CA. Then $\bigcup_{f \in \mathcal{F}} L(\{f\}) \subset L(\mathcal{F})$. If the automata in $\mathcal{F}$ commute, equality holds.*

Proof. The first claim is clear from the definition. Suppose then that the automata commute and $x \in L(\mathcal{F})$, so that there is a sequence $(f_i)_{i \in \mathbb{N}}$ over $\mathcal{F}$ such that for all n, $f_n^{-1}(\cdots f_1^{-1}(x) \cdots) \neq \emptyset$. One of the automata, say f, must occur infinitely many times in the sequence. Let $i \in \mathbb{N}$. Since the automata commute, we can move i copies of f to the beginning of the sequence, so that $f^{-i}(x) \neq \emptyset$.

Since limit sets are always nonempty, the question whether intersections of limit sets are always limit sets themselves is trivially false. In the case of nonempty intersections, we have the following counterexamples:

Example 2. The class $\infty-\mathrm{LIM}_u$ is not closed under nonempty intersection.

Proof. Take two subshifts over the alphabet $\{0, 1, 2\}$: X is a one-step SFT with the forbidden words $\{10, 11, 02, 22\}$, and Y is the sofic shift with forbidden words $\{20^n1 \mid n \in \mathbb{N}\}$. By Lemma 8 X is the stable limit set of some CA, and clearly Y is the unstable limit set of the CA that moves 1's to the right and 2's to the left, destroying them when they collide.

Consider then the shifts $Z = X \times Y$ and $Z' = Y \times X$. Now both Z and Z' are unstable limit sets of the product automata. However, their intersection, which is the product with itself of the orbit closure of ${}^\infty0120^\infty$, can't be the limit set of any family of CA by Corollary 1.

The following example can also be found in [3].

Example 3. The class $\infty-\mathrm{LIM}_s$ is not closed under nonempty intersection.

Proof. Let f be the radius $\frac{1}{2}$ CA over the alphabet $S = \{0, 1, 2\}$ whose local rule is given by the following table:

	0	1	2
0	0	1	1
1	2	0	0
2	2	0	0

The automaton f marks every transition $0x$ with 1 and $x0$ with 2, where $x \in \{1, 2\}$, and otherwise produces 0. Denote by X the subshift with forbidden words $\{a0^na \mid a \in \{1, 2\}\}$. It is clear that $L_1(\{f\}) \subset X$. On the other hand, if $x \in X$, we can easily construct a preimage $y \in X$ for it: Note that when we consider 1 and 2 equal, f behaves like the binary XOR automaton with radius $\frac{1}{2}$. Since the XOR automaton is surjective, we obtain an f-preimage for y by taking a suitable XOR-preimage for it, and replacing every other 1 by 2. This shows that $X \subset f(X)$, from which it follows that $L(\{f\}) = X$.

Let g be the symbol-transforming automaton that maps $0 \mapsto 0$, $1 \mapsto 1$ and $2 \mapsto 0$ having the stable limit set $\{0, 1\}^{\mathbb{Z}}$. Now, the intersection of the limit sets of f and g is the orbit closure of ${}^\infty010^\infty$, which, again by Corollary 1, is not the limit set of any family of automata.

In fact, the previous proofs show that $1-\mathrm{LIM}_x$ is not closed under intersection for $x \in \{s, u\}$.

Question 2. Is $\infty-\mathrm{LIM}_u$ closed under union?

Another interesting question is whether elements of $k-\mathrm{LIM}_x$ can be decomposed into finite unions of elements of $m-\mathrm{LIM}_x$, for $m < k$. In the stable case, the following theorem proves this in the positive for $m = 2$, but the case $m = 1$ is still unknown. In the unstable case, nothing is known.

Theorem 8. *If $X \in \infty-\mathrm{LIM}_s$, then X is the union of a finite number of subshifts in $2-\mathrm{LIM}_s$.*

Proof. We may assume X is the limit set of a family $\mathcal{F} = \{f_1, \dots, f_k\}$ with $X = L_1(\mathcal{G})$. Let $X_i = f_i(S^{\mathbb{Z}})$, noting that $X = \bigcup_i X_i$. Consider one of the (mixing) components X_i. Without loss of generality, we may assume that $X_i \subsetneq X_j$ for no $i \neq j$: Suppose that such i and j exist. We claim that we can remove f_i from $\mathcal{F}$ without changing the limit set. Clearly, the limit set can't grow, so it suffices to show that $X_i \subset f(X)$ for some $f \in \mathcal{F} - \{f_i\}$. But this is clear, since some point of X_j is doubly transitive and must have a preimage in X with some $f \in \mathcal{F}$, and necessarily $f \neq f_i$.

Let $x \in X_i$ be doubly transitive, and let $f \in \mathcal{F}$ be such that $f(y) = x$, where $y \in X_{j_1}$ for some j_1. Now we again see that f maps X_{j_1} onto X_i, so $f = f_i$. We repeat this argument for X_{j_1} to obtain j_2 such that $f_{j_1}(X_{j_2}) = X_{j_1}$, and continue inductively to obtain a sequence $(j_n) \in [1,k]^{\mathbb{N}}$ such that $f_{j_n}(X_{j_{n+1}}) = X_{j_n}$ for all n. Since $[1,k]$ is finite, we have that $j_m = j_{m+p}$ for some m and $p > 0$.

Now denote $Y = X_i$ and $Z = X_{j_m}$, and consider the CA

$$g = f_i \circ f_{j_1} \circ \dots \circ f_{j_{m-1}}$$

and

$$h = f_{j_m} \circ \dots \circ f_{j_{m+p-1}}.$$

We clearly have $L(\{g,h\}) \subset X$, since g and h are compositions of automata in $\mathcal{F}$. Since $h(S^{\mathbb{Z}}) = h(Z) = Z$ and $g(S^{\mathbb{Z}}) = g(Z) = Y$, we have that $Y \subset L(\{g,h\}) = Y \cup Z$. Also, the limit set is reached in one step. Now we have proved that $X_i \subset L(\{g,h\}) \subset X$ and $L(\{g,h\}) \in 2{-}\mathrm{LIM}_s$, which completes the proof since i was arbitrary.

Question 3. Is there a subshift $X \in 2{-}\mathrm{LIM}_s$ which is not a union of a finite number of subshifts in $1{-}\mathrm{LIM}_s$?

6 Acknowledgements

We'd like to thank Pierre Guillon for Lemma 12 and for pointing us in the direction of [7].

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