

No Tits alternative for cellular automata

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Abstract

We show that the automorphism group of a one-dimensional full shift (the group of reversible cellular automata) does not satisfy the Tits alternative. That is, we construct a finitely generated subgroup which is not virtually solvable yet does not contain a free group on two generators. We give constructions both in the two-sided case (acting group \mathbb{Z}) and the one-sided case (acting monoid \mathbb{N} , alphabet size at least eight). Lack of Tits alternative follows for several groups of symbolic (dynamical) origin: automorphism groups of two-sided one-dimensional uncountable sofic shifts, automorphism groups of multidimensional subshifts of finite type with positive entropy and dense minimal points, automorphism groups of full shifts over non-periodic groups, and the mapping class groups of two-sided one-dimensional transitive SFTs. We also show that the classical Tits alternative applies to one-dimensional (multi-track) reversible linear cellular automata over a finite field.

1 Introduction

In [49] Jacques Tits proved that if F is a field (with no restrictions on characteristic), then a finitely generated subgroup of $GL(n, F)$ either contains a free group on two generators or contains a solvable subgroup of finite index. We say that a group G *satisfies the Tits alternative* if whenever H is a finitely generated subgroup of G , either H is virtually solvable or H contains a free group with two generators. Whether an infinite group satisfies the Tits alternative is one of the natural first questions to ask.

The fact that $GL(n, F)$ satisfies the Tits alternative implies several things:

- The ‘Von Neumann conjecture’, that a group is amenable if and only if it contains no nonabelian free subgroup, is true for linear groups.¹
- Linear groups cannot have intermediate growth. Generally known as the *Milnor problem* [41].
- Linear groups have no infinite finitely generated periodic² subgroups. Generally known as the *Burnside problem* [14].

¹Mentioned open by Day in [23].

²A group G is *periodic*, or *torsion*, if $\forall g \in G : \exists n \geq 1 : g^n = 1$.

The first item is true because solvable groups are amenable. The second is true by the theorem of Milnor [41] and Wolf [52], which states that if G is finitely generated and solvable then either G is virtually nilpotent or G has exponential growth rate. The third is true because free groups are not periodic, and solvable groups cannot have finitely generated infinite periodic subgroups because locally finite groups are closed under group extensions.

These three properties (or lack thereof) are of much interest in group theory, since in each case whether groups can have these ‘pathological properties’ was open for a long time. It seems that none of the three have been answered for automorphism groups of full shifts (equivalently, groups of reversible cellular automata). We show that the classical Tits alternative is not enough to solve the three questions listed – it is not true. Concretely, we show that there is a residually finite variant of $A_5 \wr \mathbb{Z}$ which does not satisfy the Tits alternative and embeds in the automorphism group of a full shift.

A (*two-sided*) *full shift* is $\Sigma^{\mathbb{Z}}$ where Σ is a finite alphabet, with dynamics of \mathbb{Z} given by the shift $\sigma : \Sigma^{\mathbb{Z}} \rightarrow \Sigma^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$. A *subshift* is a topologically closed shift-invariant subsystem of a full shift. A special case is a *sofic shift*, a subsystem of a full shift obtained by forbidding a regular language of words, and *SFTs* (subshift of finite type) are obtained by forbidding a finite language. An *endomorphism* of a subshift is a continuous self-map of it, which commutes with the shift. The *automorphism group* of a subshift X , denoted by $\text{Aut}(X)$, is the group of endomorphisms having (left and right) inverses, and the automorphism group of a full shift is also known as the *group of reversible cellular automata*. See [38] for precise definitions, [34] for definitions in the multidimensional case, and [16] for subshifts on general groups. All these notions have *one-sided* versions where \mathbb{N} is used in place of \mathbb{Z} . In the case of one-sided subshifts we will only discuss full shifts $\Sigma^{\mathbb{N}}$.

As far as symbolic dynamics goes, automorphism groups of subshifts are a classical topic [33, 19], with lots of progress in the 80’s and 90’s [12, 13, 37, 10] especially in sofic settings, but also in others [35, 27]. In the last few (half a dozen) years there has been a lot of interest in these groups [43, 48, 44, 21, 18, 22, 26, 25, 20] especially in settings where the automorphism group is, for one reason or another, more restricted. Also the full shift/sofic setting, which we concentrate on in this paper, has been studied in recent works [28, 45, 46, 3].

The popular opinion is that the automorphism group of a full shift is a complicated and intractable object. However, with the Tits alternative (or the three questions listed above) in mind, looking at known (published) finitely generated subgroups as purely abstract groups does not really support this belief. As far as the author knows, all that is known about the set of subgroups of $\text{Aut}(\Sigma^{\mathbb{Z}})$ for a nontrivial alphabet Σ follow from the facts that it is independent of the alphabet, contains the right-angled Artin groups (graph groups) and is closed under direct and free products (‘cograph products’) and finite group extensions. See [13, 37, 45]. The author does not know whether all groups generated by these facts satisfy the Tits alternative,³ but believes that if finite group extensions are replaced with just containment of finite groups, this does follow from the results of [2].

Some of the known (families of) groups which satisfy the Tits alternative are

³More precisely, I do not know how finite extensions play together with graph products, when the operations are alternated.

hyperbolic groups [31], outer automorphism groups of free groups [5], finitely-generated Coxeter groups [42], and right-angled Artin groups and more generally groups obtained by graph products from other groups satisfying the Tits alternative, under minor restrictions [2]. In particular, we obtain that the automorphism group of a full shift contains a finitely-generated subgroup which is not embeddable in any such group.

Two particularly famous concrete examples of groups that do not satisfy the Tits alternative are the Grigorchuk group [30] and Thompson’s group F [15]. These groups also have many other interesting properties, so it would be more interesting to embed them instead of inventing a new group for the task. The Grigorchuk group can indeed be embedded in the automorphism group of a subshift, by adapting the construction in [7], but the author does not know whether it embeds in the automorphism group of a sofic shift. Thompson’s group F embeds in the automorphism group of an SFT [47], but is not residually finite, and thus does not embed in the automorphism group of a full shift. We mention also that there are solvable groups of derived length three whose automorphism groups do not satisfy the alternative [32].

The variant of $A_5 \wr \mathbb{Z}$ we describe is not a very complex group, and it is plausible that some weaker variant of the Tits alternative holds in automorphism groups of full shifts, and allows this type of groups in place of ‘virtually solvable groups’. In particular, our group is elementarily amenable [17], and one could ask whether every finitely generated subgroup of the automorphism group of a mixing SFT is either elementarily amenable or contains a free nonabelian subgroup. If this were the case, it would solve the Von Neumann, Milnor and Burnside problems for automorphism groups of mixing SFTs.

The group we construct satisfies the law $[g, h]^{30}$, and is thus an example of a residually finite group which satisfies a law, but does not satisfy the Tits alternative. It turns out that such an example has been found previously [24, Theorem 1], and we were delighted to find that indeed our example sits precisely in the variety used in their theorem. However, our example is rather based on an answer⁴ of Ian Agol on the mathoverflow website [36]. The idea behind the embedding is based on Turing machines [3] in the two-sided case. In the one-sided case we use a commutator trick applied to subshifts acting on finite sets.

2 Results and corollaries

In the two-sided case, we obtain several results, all based on the same construction (Lemma 3) and the fact the automorphism group of the full shift embeds quite universally into automorphism groups of subshifts.

In the statements, Σ and A are arbitrary finite alphabets.

Theorem 1. *For any alphabet Σ , the group $\text{Aut}(\Sigma^{\mathbb{Z}})$ of reversible cellular automata on the alphabet Σ does not satisfy the Tits alternative.*

The following summarizes the (well-known) embeddings listed in Section 5, and the corollaries for the Tits alternative.

⁴The answer reads ‘ $A_5 \wr \mathbb{Z}$ ’.

Theorem 3. *Let G be an infinite finitely-generated group, and $X \subset \Sigma^G$ a subshift. Then we have $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$, and thus $\text{Aut}(X)$ does not satisfy the Tits alternative, if one of the following holds:*

- $G = \mathbb{Z}$ and X is an uncountable sofic shift,
- $G = \mathbb{Z}^d$ and X is a subshift of finite type with positive entropy and dense minimal points.
- G is not periodic and X is a nontrivial full shift.

The first embedding is from [37, 45], the second is from [34].

The *mapping class group* of a subshift is defined in [9]. Combining the embedding theorem of [37, 45] and results of [9] (and a short additional argument) gives the following.

Theorem 2. *If $X \subset \Sigma^{\mathbb{Z}}$ is a nontrivial transitive SFT, then $\text{Aut}(A^{\mathbb{Z}}) \leq \mathcal{M}_X$, and thus \mathcal{M}_X does not satisfy the Tits alternative.*

Automorphisms of two-sided full shifts also appear in some less obviously symbolic dynamical context, in particular the *rational group* \mathcal{Q} of [29] contains the automorphism group of a full shift, [51] constructs its classifying space (more generally those of mixing SFTs), implementing it as the fundamental group of a simplicial complex built out of integer matrices, and [50] ‘realizes’⁵ automorphisms of full shifts in the centralizer of a particular homeomorphism of a sphere of any dimension at least 5.

In the case of one-sided SFTs, there are nontrivial restrictions on finite subgroups, and the group is generated by elements of finite order. The automorphism group of $\Sigma^{\mathbb{N}}$ is not finitely generated if $|\Sigma| \geq 3$, while $|\text{Aut}(\{0, 1\}^{\mathbb{N}})| = 2$ [33, 11]. We prove that the Tits alternative also fails in the one-sided case.

Theorem 4. *Let $|\Sigma| \geq 8$. Then $\text{Aut}(\Sigma^{\mathbb{N}})$ does not satisfy the Tits alternative.*

Of course, this group embeds in $\text{Aut}(\Sigma^{\mathbb{Z}})$, so Theorem 1 follows from Theorem 4. While $\text{Aut}(\Sigma^{\mathbb{Z}})$ embeds in many situations where we have a symbolic group action, $\text{Aut}(\Sigma^{\mathbb{N}})$ embeds more generally in many monoid and (non-vertex-transitive) graph contexts, though unlike in the case of \mathbb{Z} , we are not aware of anything concrete to cite.

The group $\text{Aut}(\Sigma^{\mathbb{N}})$ also arises in a less obviously symbolic context: It is shown in [6] that if S_d denotes the set of centered (coefficient of x^{d-1} is zero) monic polynomials p of degree d over the complex numbers, such that the filled-in Julia set (points whose iteration stays bounded) of p does not contain any critical point, then $\pi_1(S_d)$ (the fundamental group of S_d as a subspace of \mathbb{C}^d) factors onto $\text{Aut}(\{0, \dots, d-1\}^{\mathbb{N}})$. Unfortunately, the Tits alternative is not closed under quotients (since free groups satisfy the Tits alternative trivially by the Nielsen-Schreier theorem), so we do not directly get a corollary for $\pi_1(S_d)$.

⁵While the precise statement in [50] is natural and interesting, it seems hard to get a non-trivial group-theoretic interpretation of this result just from the statement of the theorem: it does follow that the automorphism group of a full shift is a subquotient of the homeomorphism group of the sphere but since this group contains free groups, so is any other countable group.

3 Residually finite wreath product

Grigorchuk group and Thompson's group F are presumably particularly famous examples of groups not satisfying the Tits alternative mostly because they are particularly famous for other reasons, and happen not to satisfy the Tits alternative – one can construct such examples directly by group extensions: $A_5 \wr \mathbb{Z}$ does not satisfy the alternative by a similar proof as that of Lemma 2.

The group $A_5 \wr \mathbb{Z}$ is not residually finite since A_5 is non-abelian, and thus cannot be embedded in the automorphism group of a full shift. Informally, there is an obvious way to 'embed' it, but this embedding is not quite a homomorphism, because the relations only hold on some 'well-formed' configurations. In this section, we describe the abstract group obtained through this 'embedding' – a kind of broken wreath product. Luckily for us, it turns out not to satisfy the Tits alternative either.

Let $N \subset \mathbb{N}$ be an infinite⁶ set, let G be a finite group and write $G \wr^N \mathbb{Z}$ for the group generated by the elements of G and a new element \circ , which act on $\bigcup_{n \in N} G^n$ by

$$a \cdot (g_1, g_2, \dots, g_n) = (ag_1, g_2, \dots, g_n)$$

for $a \in G$ and $n \in N$, and

$$\circ \cdot (g_1, g_2, \dots, g_n) = (g_2, g_3, \dots, g_n, g_1).$$

More precisely, the formulas attach a bijection on $\bigcup_{n \in N} G^n$ to each $a \in G$ and to \circ (by the above formulas), and $G \wr^N \mathbb{Z}$ is (up to isomorphism) the group of bijections they generate. This is a variant of the usual wreath product of G and \mathbb{Z} , but $G \wr^N \mathbb{Z}$ is obviously residually finite for any finite group G , since it is defined by its action on the finite sets G^n . Note that \circ simply rotates (the coordinates of) G^n for $n \in N$, and generates a copy of \mathbb{Z} .

A *subquotient* of a group is a factor of a subgroup.

Lemma 1. *Let G be a finitely generated group which has A_5^n as a subquotient for infinitely many n . Then G is not virtually solvable.*

Proof. Suppose it is. Then there is an index k solvable subgroup $K \leq G$. Let $G_1 \leq G$ be a subgroup. Then $[G_1 : K \cap G_1] \leq k$ and K is solvable. Let $h : G_1 \rightarrow G_2$ be a surjective homomorphism. Then for $K' = h(K \cap G_1)$, we have $[G_2 : K'] \leq [G_1 : K \cap G_1] \leq k$. Because K' is a subquotient of the solvable group K , it is solvable, and we obtain that any subquotient of G has a solvable subgroup of index at most k .

The group A_5^n is now a subquotient of G for arbitrarily large (thus all) n , so A_5^n has a solvable subgroup K' of index at most k . We claim that if n is sufficiently large with respect to k , this is a contradiction. A simple argument for $n \geq k$ being enough is that if there is some coordinate i such that K' projected to that coordinate is surjective, then the non-solvable group A_5 is a quotient of K' , contradicting solvability of K' . If on the other hand in every coordinate the projection of K' is a proper subgroup of A_5 , then there are at least $n + 1$ cosets of K' in A_5^n . (Actually there are exponentially many cosets in n .) \square

⁶The definition works fine if N is a finite set, but then the group we define is finite, and thus satisfies the alternative.

Lemma 2. *If $N \subset \mathbb{N}$ is infinite, then the group $A_5 \int^N \mathbb{Z}$ is not virtually solvable and does not contain a free nonabelian group.*

Proof. First observe that $\phi(a) = 0$ for $a \in G$ and $\phi(\circlearrowleft) = 1$ extends to a well-defined homomorphism $\phi : A_5 \int^N \mathbb{Z} \rightarrow \mathbb{Z}$ (where we use that N is infinite).

Suppose first that the free group $F_2 \leq A_5 \int^N \mathbb{Z}$ is generated freely by g, h , and $N \subset \mathbb{N}$ is arbitrary. Clearly on any A_5^n , $[g, h]$ performs zero total rotation, that is, $\phi([g, h]) = 0$. Thus independently of n , we have $[g, h]^{30} \cdot \vec{v} = \vec{v}$ for any n and $\vec{v} \in A_5^n$, since the exponent of A_5 is 30. This implies that g and h satisfy a nontrivial relation, contradicting the assumption.⁷

So show that $A_5 \int^N \mathbb{Z}$ is not virtually solvable, we need to show that it has A_5^n as a subquotient for arbitrarily large n . We show it is a subgroup of a factor (thus also a factor of a subgroup). Pick $n \in N$. Then $A_5 \int^N \mathbb{Z}$ acts on A_5^n and a moment's reflection shows that this induces a surjective homomorphism from $A_5 \wr \mathbb{Z}/n\mathbb{Z}$ to $A_5 \wr \mathbb{Z}/n\mathbb{Z}$. We have $A_5^n \leq A_5 \wr \mathbb{Z}/n\mathbb{Z}$. \square

Since A_5 acts faithfully on $\{1, 2, 3, 4, 5\}$, it is easy to see that the following group (again defined by its action) is isomorphic to $A_5 \int^N \mathbb{Z}$: Let elements of A_5 and \circlearrowleft act on $\bigcup_{n \in N} \{1, 2, 3, 4, 5\}^n$ by

$$a \cdot (m_1, m_2, \dots, m_n) = (a(m_1), m_2, \dots, m_n)$$

for $a \in A_5$, and

$$\circlearrowleft \cdot (m_1, m_2, \dots, m_n) = (m_2, m_3, \dots, m_n, m_1).$$

4 The construction in the two-sided case

If Σ is a finite alphabet, write Σ^* for the set of words over Σ (including the empty word).

Lemma 3. *There exists an alphabet Σ such that we have $A_5 \int^{2\mathbb{N}} \mathbb{Z} \leq \text{Aut}(\Sigma^{\mathbb{Z}})$.*

Proof. Let $A = \{1, 2, 3, 4, 5\}$ and choose $\Sigma = \{\#\} \cup A^2$. Before describing the automorphism, we define an auxiliary faithful action of $A_5 \int^{2\mathbb{N}} \mathbb{Z}$ on finite words. Think of a word $w \in (A^2)^n$ as two words $u, v \in A^n$ on top of each other, the topmost one defined by $u_i = (w_i)_1$ and the second $v_i = (w_i)_2$, for $i = 1, 2, \dots, n$. We use the notation $w = \begin{bmatrix} u \\ v \end{bmatrix} = ([u_1], [u_2], \dots, [u_n])$. Define a bijection $\psi : (A^2)^n \rightarrow A^{2n}$ by $\psi(\begin{bmatrix} u \\ v \end{bmatrix}) = uv^R$ where v^R is the reversal of v defined by $a^R = a$ for $a \in A$ and $(va)^R = a(v^R)$.

Now, conjugate the defining action of $A_5 \int^{2\mathbb{N}} \mathbb{Z}$ on A^{2n} to $(A^2)^n$ through ψ to obtain the action

$$a \cdot ([\begin{smallmatrix} m_1 \\ m'_1 \end{smallmatrix}], [\begin{smallmatrix} m_2 \\ m'_2 \end{smallmatrix}], \dots, [\begin{smallmatrix} m_n \\ m'_n \end{smallmatrix}]) = ([\begin{smallmatrix} a \cdot m_1 \\ m'_1 \end{smallmatrix}], [\begin{smallmatrix} m_2 \\ m'_2 \end{smallmatrix}], \dots, [\begin{smallmatrix} m_n \\ m'_n \end{smallmatrix}])$$

for $a \in A_5$, and for \circlearrowleft the following counter-clockwise ‘conveyor belt’ rotation

$$\circlearrowleft \cdot ([\begin{smallmatrix} m_1 \\ m'_1 \end{smallmatrix}], [\begin{smallmatrix} m_2 \\ m'_2 \end{smallmatrix}], \dots, [\begin{smallmatrix} m_n \\ m'_n \end{smallmatrix}]) = ([\begin{smallmatrix} m_2 \\ m'_1 \end{smallmatrix}], [\begin{smallmatrix} m_3 \\ m'_1 \end{smallmatrix}], [\begin{smallmatrix} m_4 \\ m'_2 \end{smallmatrix}], \dots, [\begin{smallmatrix} m'_n \\ m'_{n-1} \end{smallmatrix}]).$$

⁷Indeed we have shown that the group satisfies a law, similarly as in [24, Theorem 1].

Now, we define our automorphisms. To $a \in A_5$ we associate the automorphism $f_a : \Sigma^{\mathbb{Z}} \rightarrow \Sigma^{\mathbb{Z}}$ defined by $f_a(x)_i = F_a(x_{i-1}, x_i)$ where $F_a : \Sigma^2 \rightarrow \Sigma$ is defined by $F_a(b, c) = c$ if $b \neq \#$, $F_a(\#, \#) = \#$ and

$$F_a(\#, \begin{bmatrix} b \\ c \end{bmatrix}) = \begin{bmatrix} a \cdot b \\ c \end{bmatrix}$$

where $a \cdot b$, is the action of $a \in A_5$ by permutation on $b \in A$. It is easy to see that f_a is an endomorphism of $\Sigma^{\mathbb{Z}}$, and $x_i = \# \iff f_a(x)_i = \#$.

To \circlearrowleft , we associate $f_{\circlearrowleft} : \Sigma^{\mathbb{Z}} \rightarrow \Sigma^{\mathbb{Z}}$ defined by $f_{\circlearrowleft}(x)_i = F_{\circlearrowleft}(x_{i-1}, x_i, x_{i+1})_i$ where $F_{\circlearrowleft} : \Sigma^3 \rightarrow \Sigma$ is defined by $F_{\circlearrowleft}(a, \#, b) = \#$ for all $a, b \in A^2$ and

$$F_{\circlearrowleft}(\#, \begin{bmatrix} c \\ d \end{bmatrix}, \#) = \begin{bmatrix} d \\ c \end{bmatrix},$$

$$F_{\circlearrowleft}(\#, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}) = \begin{bmatrix} e \\ c \end{bmatrix},$$

$$F_{\circlearrowleft}(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \#) = \begin{bmatrix} d \\ b \end{bmatrix},$$

$$F_{\circlearrowleft}(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}) = \begin{bmatrix} e \\ b \end{bmatrix}$$

for all $a, b, c, d, e, f \in A$. It is easy to see that F_{\circlearrowleft} is also an endomorphism of $\Sigma^{\mathbb{Z}}$, and $x_i = \# \iff F_{\circlearrowleft}(x)_i = \#$.

Now, let $Y \subset \Sigma^{\mathbb{Z}}$ be the set of points x where both the left tail $x_{(-\infty, -1]}$ and the right tail $x_{[0, \infty)}$ contain infinitely many $\#$ -symbols, and consider any point $x \in Y$. Then x splits uniquely into an infinite concatenation of words

$$x = \dots w_{-2} \# w_{-1} \# w_0 \# w_1 \# w_2 \dots$$

where $w_i \in (A^2)^*$ for all $i \in \mathbb{Z}$. If f is either f_{\circlearrowleft} or one of the f_a for $a \in A_5$, then the decomposition of $f(x)$ contains $\#$ s in the same positions as that of x , in the sense that (up to shifting indices)

$$f(x) = \dots u_{-2} \# u_{-1} \# u_0 \# u_1 \# u_2 \dots$$

where $|u_i| = |w_i|$ for all i and the words begin in the same coordinates. Thus $f(x) \in Y$. It is easy to see that between two $\#$ s, the mapping $w_i \mapsto u_i$ performed by f is precisely the one we defined for words in $(A^2)^*$ described above for the corresponding generator of $A_5 \int^{2\mathbb{N}} \mathbb{Z}$.

It follows that $a \mapsto f_a|_Y$, $\circlearrowleft \mapsto f_{\circlearrowleft}|_Y$ extends uniquely to an embedding of $A_5 \int^{2\mathbb{N}} \mathbb{Z}$ into the group of self-homeomorphisms of Y . Since Y is dense in $\Sigma^{\mathbb{Z}}$ and f_a and f_{\circlearrowleft} are endomorphisms of $\Sigma^{\mathbb{Z}}$, $a \mapsto f_a$, $\circlearrowleft \mapsto f_{\circlearrowleft}$ extends to an embedding of $A_5 \int^{2\mathbb{N}} \mathbb{Z}$ into $\text{Aut}(\Sigma^{\mathbb{Z}})$. \square

5 Embedding results

In this section we list some embeddings from the literature. We start with uncountable sofic shifts, where uncountable refers to the cardinality of the set of points. Note that full shifts are uncountable sofic shifts ($\Sigma^{\mathbb{N}}$ is uncountable, and the empty language is regular).

The following is [45, Lemma 7].

Lemma 4. *If $X \subset \Sigma^{\mathbb{Z}}$ is an uncountable sofic shift, then $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$ for any alphabet A .*

Proposition 1. *If X is an uncountable sofic shift, then $A_5 \int^{\mathbb{N}} \mathbb{Z} \leq \text{Aut}(X)$.*

Proof. By Lemma 4, we have $\text{Aut}(\Sigma^{\mathbb{Z}}) \leq \text{Aut}(X)$ where Σ is the alphabet of Lemma 3. We have $A_5 \int^{2\mathbb{N}} \mathbb{Z} \leq \text{Aut}(\Sigma^{\mathbb{Z}})$ by Lemma 3, so it is enough to check that $A_5 \int^{\mathbb{N}} \mathbb{Z} \leq A_5 \int^{2\mathbb{N}} \mathbb{Z}$. One can check that such an embedding is induced by $a \mapsto a$ for $a \in A_5$, and $\circ \mapsto \circ^2$. \square

As for countable sofic shifts, we do not have a characterization of the situations when the automorphism group satisfies the Tits alternative. However, in that setting, there are stronger methods for studying the three embeddability questions listed in the introduction and we refer to [47].

The following is [34, Theorem 3].

Lemma 5. *If $X \subset \Sigma^{\mathbb{Z}^d}$ is an SFT with positive entropy and dense minimal points, then $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$ for any alphabet A .*

Below, minimal points are points whose orbit-closure is minimal as a dynamical system.

Theorem 3. *Let G be an infinite finitely-generated group, and $X \subset \Sigma^G$ a subshift. Then we have $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$, and thus $\text{Aut}(X)$ does not satisfy the Tits alternative, if one of the following holds:*

- $G = \mathbb{Z}$, X is an uncountable sofic shift,
- $G = \mathbb{Z}^d$, X is an SFT with positive entropy and dense minimal points, or
- G is not periodic, X is a nontrivial full shift.

Proof. The first two were proved above. The third item is true, because if $\mathbb{Z} \leq G$ by $n \mapsto g^n$ and $X = A^G$ for $A \subset \Sigma$, then if K is a set of left cosets representatives for $\langle g \rangle$ then $f \in \text{Aut}(A^{\mathbb{Z}})$ directly turns into an automorphism of A^G by $\hat{f}(x)_{kg^n} = f(y)_n$ where $y_i = x_{kg^i}$ and $k \in K$. \square

The first item generalizes to sofic $H \times \mathbb{Z}$ -shifts where H is finite by [46], and could presumably be generalized to virtually-cyclic groups with the same idea. By symmetry-breaking arguments, we believe the third item generalizes to SFTs with a nontrivial point of finite support, on any group which is not locally finite, and also to cellular automata acting on sets of colorings of undirected graphs, but this is beyond the scope of the present paper. A generalization of the second item seems worth conjecturing more explicitly.

Conjecture 1. *Let G be an amenable group which is not locally finite and X a subshift of finite type with positive entropy and dense minimal points. Then we have $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$ for any finite alphabet A .*

Next, we deal with the mapping class group. By definition, the group $\langle \sigma \rangle$ is contained in the center of $\text{Aut}(X)$ for any subshift X , in particular this is a normal subgroup.

Lemma 6. *Let X be any uncountable sofic shift. Then $\text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)/\langle \sigma \rangle$ for every finite alphabet A .*

Proof. Let $\phi : \text{Aut}(A^{\mathbb{Z}}) \leq \text{Aut}(X)$ be the embedding given by Lemma 4. From its proof in [45], it is easy to see that there exists an infinite subshift $Y \leq X$ such that $\phi(f)$ fixes every point in Y for every $f \in \text{Aut}(A^{\mathbb{Z}})$; namely the maps $\phi(f)$ only act nontrivially at a bounded distance from an unbordered word w which can taken to be arbitrarily long.

We show that based on only this, ϕ is automatically also an embedding of $\text{Aut}(A^{\mathbb{Z}})$ into $\text{Aut}(X)/\langle\sigma\rangle$. Suppose not, and that $\phi(f) \circ \sigma^k = \phi(g)$ for some $f, g \in \text{Aut}(A^{\mathbb{Z}})$. Then in particular $\phi(f) \circ \sigma^k(y) = \phi(g)(y) \implies \sigma^k(y) = y$ for every $y \in Y$. If $k \neq 0$ this is a contradiction since Y is an infinite subshift. If $k = 0$, then $\phi(f) = \phi(g)$ implies $f = g$ since $\phi : \text{Aut}(A^{\mathbb{Z}}) \rightarrow \text{Aut}(X)$ is an embedding. \square

The following is now a straightforward corollary of [9]. See [9] for the definition of the *mapping class group* \mathcal{M}_X of a subshift X (which is not needed in the proof).

Lemma 7. *Let X be any transitive SFT. Then $\text{Aut}(A^{\mathbb{Z}}) \leq \mathcal{M}_X$ for every alphabet A .*

Proof. In [9, Theorem 5.6], it is shown in particular that if X is a transitive SFT, then its mapping class group contains an isomorphic copy of $\text{Aut}(X)/\langle\sigma\rangle$, which then contains a copy of $\text{Aut}(A^{\mathbb{Z}})$ by the above lemma. \square

In [29, Corollary 5.5], it is shown that the automorphism group of every full shift embeds in the group \mathcal{Q} that they define, sometimes called the *rational group*. Thus we also obtain a new proof that \mathcal{Q} does not satisfy the Tits alternative.

6 One-sided automorphism groups

The automorphism group of the full shift $\{0, 1\}^{\mathbb{N}}$ is isomorphic to \mathbb{Z}_2 . For large enough alphabets, however, we show that $\text{Aut}(\Sigma^{\mathbb{N}})$ does not satisfy the Tits alternative. This gives another proof of Theorem 1.

The high-level idea of the proof is that we can associate to a subshift a kind of action of it on a finite set, in a natural way. Mapping $n \mapsto (0^{n-1}1)^{\mathbb{Z}}$, the action of the subshift generated by the image of $N \subset \mathbb{N}$ corresponds to the group $A_5 \int^N \mathbb{Z}$ defined previously. It turns out that Lemma 2 generalizes to such actions, and any infinite subshift can be used in place of this (almost) periodic subshift. The generalization is based on a commutator trick from [4, 40, 1, 8, 3]. The trick to adapting the construction to cellular automata on \mathbb{N} is to consider ‘actions of the trace of another cellular automaton’.

We only give the definitions in the special case of A_5 and binary subshifts. Let $X \subset \{0, 1\}^{\mathbb{Z}}$ be a subshift and let $Y = \{1, 2, 3, 4, 5\}$. For each $g \in A_5$ and $j \in \{0, 1\}$ define a bijection $g_j : Y \times X \rightarrow Y \times X$ by $g_j(y, x) = (g \cdot y, x)$ for $x_0 = j$ and $g_j(y, x) = (y, x)$ for $x_0 \neq j$. Define a bijection $\circlearrowleft \curvearrowright Y \times X$ by $\circlearrowleft(y, x) = (y, \sigma(x))$. Denote the group generated by these maps by $A_5 \int^X \mathbb{Z}$.

Lemma 8. *Let $X \subset \{0, 1\}^{\mathbb{Z}}$ be infinite. Then the group $A_5 \int^X \mathbb{Z}$ is not virtually solvable and does not contain a free nonabelian group.*

Proof. Observe that $\phi(g) = 0$ for $g \in A_5$ and $\phi(\circlearrowleft) = 1$ extends to a well-defined homomorphism $\phi : A_5 \int^X \mathbb{Z} \rightarrow \mathbb{Z}$ (since X is infinite). Suppose first that $F_2 \leq A_5 \int^N \mathbb{Z}$ is generated freely by g, h . Then as in Lemma 2, we get $\phi([g, h]) = 0$, and again we have $[g, h]^{30} \cdot (y, x) = (y, x)$.

We now show that A_5^n is a subgroup of $A_5 \int^X \mathbb{Z}$ for arbitrarily large n , and the claim follows from Lemma 1.

Consider the cylinder sets $[w]_i = \{x \in \{0, 1\}^{\mathbb{Z}} \mid x_{[i, i+|w|-1]} = w\}$ and for $g \in A_5$, $w \in \{0, 1\}^*$ and $i \in \mathbb{Z}$ define

$$\pi_{g,i,w}(y, x) = (gy, x) \text{ if } x \in [w]_i, \text{ and } \pi_{g,i,w}(y, x) = (y, x) \text{ otherwise.}$$

We claim that $\pi_{g,i,w} \in A_5 \int^X \mathbb{Z}$ for all g, i, w . To see this, observe that by the definition of how $g_0, g_1 \in A_5$ act on $Y \times X$, and by conjugating with \circlearrowleft , we have $\pi_{g,w,i} \in A_5 \int^X \mathbb{Z}$ for all $g \in A_5, i \in \mathbb{Z}$ and $w \in \{0, 1\}$. We proceed inductively on $|w|$: Let $w = uv$ where $u, v \in \{0, 1\}^+$ are nonempty words, and let $a \in A_5$ be any commutator, that is, $a = [b, c]$. Then one sees easily that

$$[\pi_{b,i,u}, \pi_{c,i+|u|,v}] = \pi_{[b,c],i,w}.$$

Because A_5 is perfect (that is, $A_5 = [A_5, A_5]$), we get that $\pi_{g,i,w} \in A_5 \int^X \mathbb{Z}$ for every $g \in A_5$. This proves the claim.

Now, let w be an unbordered word⁸ of length n [39] that occurs in some point of X . Then the elements $\pi_{g,i,w}$ for $0 \leq i < |w|$ generate a group isomorphic to A_5^n (because the fact w is unbordered implies the supports of their actions are disjoint), and we conclude. \square

Theorem 4. *Let $|\Sigma| \geq 8$. Then $\text{Aut}(\Sigma^{\mathbb{N}})$ does not satisfy the Tits alternative.*

Proof. Let $\Sigma = Y \sqcup A \sqcup B$ where A, B are finite sets, $|A| \geq 3$ and $|Y| = 5$.

Let $f : \Sigma^{\mathbb{N}} \rightarrow \Sigma^{\mathbb{N}}$ be any reversible cellular automaton of infinite order such that $x_i \in A \iff f(x)_i \in A$ and $x_i \notin A \implies f(x)_i = x_i$. One construction is the following: for $a, b \in A$, $a \neq b$, define $F_{a,b} : \Sigma^{\mathbb{Z}} \rightarrow \Sigma^{\mathbb{Z}}$ by

$$\text{for all } c \notin \{a, b\} : F_{a,b}(a, c) = b \text{ and } F_{a,b}(b, c) = a$$

and $F_{a,b}(c, d) = c$ otherwise. Define $f_{a,b} \in \text{Aut}(\Sigma^{\mathbb{N}})$ by $f_{a,b}(x)_0 = F_{a,b}(x_0, x_1)$. This is an involution, so it is an automorphism. Now it is easy to check that for any distinct elements $a, b, c \in A$, $f_{a,b} \circ f_{b,c}$ is of infinite order, by considering its action on any point that begins with the word $b^k a$.

The *trace* of f is the subshift of $\Sigma^{\mathbb{Z}}$ consisting of y such that for some $x \in \Sigma^{\mathbb{N}}$, $f^n(x)_0 = y_n$ for all n . Since f is of infinite order on $\Sigma^{\mathbb{N}}$ and fixes all symbols not in A , its trace T_f intersected with $A^{\mathbb{Z}}$ is infinite. Let $\pi : A \rightarrow \{0, 1\}$ be a function, and extend it to $\pi : A^{\mathbb{Z}} \rightarrow \{0, 1\}^{\mathbb{Z}}$ in the obvious way. Pick π so that the subshift $X = \pi(T_f \cap A^{\mathbb{Z}})$ is infinite. This is possible because if the indicator function of some symbol gives a finite subshift, then this symbol appears with a fixed period in every point, so if all indicator functions give finite subshifts, every point can be described by a finite amount of data.

For each permutation $p \in A_5$ and $i \in \{0, 1\}$, take the $f_{p,i}$ to be the automorphism defined by $f_{p,i}(x)_0 = p(x_0)$ if $x_0 \in Y$, $x_1 \in A$ and $\pi(x_1) = i$, and by $f_{p,i}(x)_0 = x_0$ otherwise.

⁸Alternatively, one can use the Marker Lemma [38] for X .

One can now check that f and $f_{p,i}$ for $p \in A_5$, $i \in \{0, 1\}$ together generate a group isomorphic to $A_5 \wr^X \mathbb{Z}$, and by the previous lemma, this group does not satisfy the Tits alternative. \square

Question 1. *For which one-dimensional transitive SFTs does the Tits alternative hold? Is it always false when the automorphism group is infinite?*

Unlike in the two-sided case, automorphism groups of one-sided full shifts do not in general embed into each other, due to restrictions on finite subgroups, see [11].

7 Tits alternative for linear cellular automata

We show that linear cellular automata, with a suitable definition, satisfy the assumptions of Tits' theorem directly.

Let K be a finite field and let V be a finite-dimensional vector space over K . Then the full shift $V^{\mathbb{Z}}$ is also a vector space over K with cellwise addition $(x+y)_i = x_i + y_i$ and cellwise scalar multiplication $(a \cdot x)_i = a \cdot x_i$. Consider the semigroup $\text{LEnd}(V^{\mathbb{Z}})$ of endomorphisms f of the full shift $V^{\mathbb{Z}}$ which are also linear maps, i.e. $f(x+y) = f(x) + f(y)$, $a \cdot f(x) = f(a \cdot x)$, and the group $\text{LAut}(V^{\mathbb{Z}}) \leq \text{LEnd}(V^{\mathbb{Z}})$ of such maps which are also automorphisms (in which case the automorphism is automatically linear as well).

The (formal) Laurent series $\sum_{i \in \mathbb{Z}} c_i \mathbf{x}^i$, where $c_i \in K$ for all i and $c_i = 0$ for all small enough i , form a field that we denote by $K((\mathbf{x}))$. By $\text{GL}(n, K((\mathbf{x})))$ we denote the group of linear automorphisms of the vector space $K((\mathbf{x}))^n$ over $K((\mathbf{x}))$. A Laurent polynomial is a Laurent series with finitely many nonzero coefficients c_i , and we write $K[\mathbf{x}, \mathbf{x}^{-1}]$ for this subring of $K((\mathbf{x}))$. In addition to being a vector space over $K((\mathbf{x}))$, $K((\mathbf{x}))^n$ is a module over $K[\mathbf{x}, \mathbf{x}^{-1}]$ and a vector space over K (by simply restricting the scalar field).

Lemma 9. *If V is an n -dimensional vector space over the finite field K , the semigroup $\text{LEnd}(V^{\mathbb{Z}})$ is isomorphic to the semigroup $M_n(K[\mathbf{x}, \mathbf{x}^{-1}])$ of n -by- n matrices over the ring $K[\mathbf{x}, \mathbf{x}^{-1}]$. The group $\text{LAut}(V^{\mathbb{Z}})$ corresponds to the invertible matrices in this isomorphism.*

Proof. We may assume $V = K^n$. Let X be the set of points $x \in (K^n)^{\mathbb{Z}}$ such that $x_{-i} = 0^n$ for all but finitely many i . Note that this is a subspace of $(K^n)^{\mathbb{Z}}$ for the K -linear structure. Define a map $\phi : X \rightarrow K[\mathbf{x}, \mathbf{x}^{-1}]^n$ by $\phi(x)_j = \sum_{i \in \mathbb{Z}} (x_i)_j \mathbf{x}^i$, and observe that this is an isomorphism of the vector spaces X and $K[\mathbf{x}, \mathbf{x}^{-1}]^n$ over K .

Suppose first that $f : (K^n)^{\mathbb{Z}} \rightarrow (K^n)^{\mathbb{Z}}$ is a linear CA, i.e. K -linear, continuous and shift-commuting. Then we see that the images of $e_i \in (K^n)^{\mathbb{Z}}$ defined by $((e_i)_0)_j = \delta_{ij}$ (where $\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ otherwise) determine f completely. If some $f(e_i)$ has infinitely many nonzero coordinates, then it is easy to see that f is not continuous at $(0^n)^{\mathbb{Z}}$. One can then directly work out the matrix representation for $\phi \circ f|_X \circ \phi^{-1}$ from the images $f(e_i)$, observing that applying the left shift σ corresponds to scalar multiplication by \mathbf{x}^{-1} . Since X is dense in $(K^n)^{\mathbb{Z}}$, f is completely determined by $f|_X$, so this representation as a matrix semigroup is faithful.

By inverting the construction, every matrix arises from the local rule of some cellular automaton. The claim about invertible matrices is straightforward. \square

The following is obvious.

Lemma 10. *The group of invertible matrices in $M_n(K[\mathbf{x}, \mathbf{x}^{-1}])$ embeds in $\mathrm{GL}(n, K((\mathbf{x})))$.*

Combining the lemmas, we get that $\mathrm{LAut}(V^{\mathbb{Z}})$ is a linear group over a finite-dimensional vector space. The result of [49] now directly applies.

Theorem 5. *Let V be a finite-dimensional vector space over a finite field K . Then the group $\mathrm{LAut}(V^{\mathbb{Z}})$ satisfies the Tits alternative.*

The author does not know if the group of (algebraic) automorphisms of a one-dimensional group shift always satisfies the Tits alternative.

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