More on wreath products of cellular automata

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Abstract

We prove that if a subgroup H of the automorphism group $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ of a non-trivial full shift acts on points of finite support with a free orbit, then for every finitely-generated abelian group A, the abstract group $A \wr H$ also embeds in $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$. The groups admitting an action with such a free orbit include $A \wr \mathbb{Z}$ for A a finite abelian group, and finitely-generated free groups. The class of such groups is also closed under commensurability and direct products. We obtain for example that $\mathbb{Z} \wr \mathbb{Z}$, $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ and $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ embed in $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$. To our knowledge, the group $\mathbb{Z} \wr \mathbb{Z}$ is the first example of a finitely-generated torsion-free subgroup of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ with infinite cohomological dimension. It answers an implicit question of Kim and Roush and an explicit question of the author. We also explore a simpler variant of the construction that gives some near-misses to iterated permutational wreath products, as well as some Neumann groups.

1 Introduction

The *full shift* is $\Sigma^{\mathbb{Z}}$, where Σ is a finite alphabet, seen as a dynamical system under the $\mathbb{Z}$-action of the shift map σ, defined by $\sigma(x)_i = x_{i+1}$. Its automorphism group $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ consists of σ-commuting self-homeomorphisms of $\Sigma^{\mathbb{Z}}$, also known as reversible cellular automata.

This paper continues the study of the family

$$\mathcal{G} = \{G \text{ group} \mid G \hookrightarrow \mathrm{Aut}(\Sigma^{\mathbb{Z}})\}$$

of groups that abstractly embed into the automorphism group of $\mathbb{Z}$-full shift, which we sometimes call *groups of cellular automata.* To our knowledge, the only "upper bounds" known on $\mathcal{G}$ is that groups in it are countable [11], residually finite [4], and the word problems of f.g. groups in $\mathcal{G}$ are in co-NP [22, 12, 4].

We know much more on the constructive side, i.e. many complicated behaviors have been exhibited in f.g. groups in $\mathcal{G}$. The Tits' alternative can fail [21], the torsion problem can be undecidable [2, 25], there can be distorted infinite cyclic subgroups [5], and the conjugacy problem is in a sense undecidable in all large enough groups in $\mathcal{G}$ [22]. We also know that $\mathcal{G}$ is closed under commensurability [12] and countable graph products [20, 23].

Less is known in the "tame" end of the spectrum of groups. In particular, very little is known about solvable groups in $\mathcal{G}$. Indeed, already abelian groups are a mystery: finitely-generated abelian group are embeddable due to closure properties of $\mathcal{G}$, but we already do not know whether the dyadic rationals $\mathbb{Z}[1/2]$ are in $\mathcal{G}$. Residual finiteness does not prevent this, and in general it is a major open problem whether $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ can have any elements of infinite order with roots of infinitely many orders [3]. For general subshifts, such behavior is exhibited in [4, 19].

Next, one may consider nilpotent groups. Now, whether there are *any* finitely-generated nilpotent groups in $\mathcal{G}$ (which are not virtually abelian) is a major open problem. The issue is that such groups always contain a copy of the three-dimensional integer Heisenberg group with presentation $\langle a, b \mid [a, [a, b]] = [b, [a, b]]\rangle$, and this group has a distorted $\mathbb{Z}$-subgroup (namely $\langle [a, b]\rangle$). We only know a single example of a distorted $\mathbb{Z}$-subgroup in the automorphism group of a $\mathbb{Z}$-subshift [5] and have no reason to believe it is the center of a Heisenberg subgroup. The question of whether the Heisenberg group is in $\mathcal{G}$ is essentially due to Kim and Roush [12].

Next, one may consider metabelian groups. Again we are in trouble already with the most basic finitely-generated examples, as the Baumslag-Solitar group $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ contains a copy of $\mathbb{Z}[1/2]$. It also has distortion elements, and here the situation is worse than with the Heisenberg group, as the distortion function is exponential (much faster than the rate demonstrated for the example in [5]).

In light of these difficult problems a more promising class to consider are wreath products. To guide us to which types of wreath products we should consider, we should consider the known restrictions: countability, residual finiteness, and the complexity of the word problem.

Countability gives no interesting restrictions. The fact $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ is residually finite, on the other hand, gives nontrivial restrictions. We recall Gruenberg's theorem [9]: The group $G = K \wr H$ is residually finite if and only if K, H are residually finite and H finite or K abelian. For H finite the wreath product embedding problem was completely solved by Kim and Roush [12], namely $\mathrm{Aut}(A^{\mathbb{Z}}) \wr H \leq \mathrm{Aut}(A^{\mathbb{Z}})$ for all such H. After taking this into account, it is easy to see that the co-NPness of the word problem does not pose further restrictions, in the sense that if A is f.g. abelian and B has co-NP word problem, then $A \wr B$ has co-NP word problem. (See Lemma 11 for the proof.)

Thus a natural question is:

Question 1. *Is $\mathcal{G}$ closed under wreath products with abelian base? More precisely, if K is an abelian group in $\mathcal{G}$ and $H \in \mathcal{G}$, do we always have $K \wr H \in \mathcal{G}$?*

Of course one can simply replace H with $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ in the above question. Before the present paper, the state-of-the-art with solvable groups obtained from wreath product constructions is [23], where we proved in particular the following theorem:

Theorem 1 ([23])**.** *For $n \geq 1$, the groups $\mathbb{Z}_2 \wr \mathbb{Z}^n$ are in $\mathcal{G}$.*

(Here $\mathbb{Z}_2$ is the group with two elements.) More generally, we constructed the groups $A \wr H$ for some groups H, including free abelian groups and free non-abelian groups, and A a finite abelian group. The cases $\mathbb{Z} \wr \mathbb{Z}$ and $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ were left open in [23]. Here, we solve these problems and slightly more:

Theorem 2. *If A is a finitely-generated abelian group, B is a finite abelian group, C is a finitely-generated free group, and $n \in \mathbb{N}$, then $A \wr ((B \wr \mathbb{Z})^n \times C^n)$ is in $\mathcal{G}$.*

The statement above is a little cluttered; it is simply a practical way to summarize (more or less) all of the (non-permutational) wreath products we were able to obtain from the main construction.

Corollary 1. *The groups $\mathbb{Z} \wr \mathbb{Z}^n$, $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$, and $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ are in $\mathcal{G}$.*

Kim and Roush wrote in 1990 [12] that all known finitely generated torsion free groups in $\mathcal{G}$ known at the time had a subgroup of finite index which is "locally nilpotent and of finite cohomological dimension". Roush clarified in private communication [18] that "locally nilpotent" should say "residually nilpotent".

Despite us nowadays knowing many examples of interesting groups in $\mathcal{G}$, to our knowledge the only known interesting finitely-generated torsion-free groups exhibited before the present paper are still precisely the graph groups from [12], which are always residually nilpotent [6, Theorem 2.3], and of finite cohomological dimension [13]. The group $\mathbb{Z} \wr \mathbb{Z}$ of course has infinite cohomological dimension, as it contains $\mathbb{Z}^d$ for all d, so our theorem provides the first counterexamples to the observation of Kim and Roush.

All torsion-free groups provided by the theorem above are residually nilpotent (see Lemma 12), so the other half of the observation of Kim and Roush remains true. There are certainly many non-(residually nilpotent) groups *with* torsion in $\mathcal{G}$, simply because $\mathcal{G}$ contains all finite groups [11]; it is easy to also find examples that are not even virtually residually finite, for example there are f.g. groups in $\mathcal{G}$ containing copies of every finite group (for example, this follows from the main result of [25]).

The methods of the present paper still do not get very far into iterated wreath products:

Question 2. *Which of $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}^2)$, $\mathbb{Z}_2 \wr (\mathbb{Z} \wr \mathbb{Z})$, $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}))$ are in $\mathcal{G}$?*

In [23], we obtained embeddings of $\mathbb{Z}_2 \wr \mathbb{Z}^n$ also in the one-sided setting (with varying alphabet sizes). The construction of the present paper does not seem to admit such a variant.

Question 3. *Are $\mathbb{Z} \wr \mathbb{Z}$, $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ and $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ groups of one-sided cellular automata, i.e. do they embed in the automorphism group of a full $\mathbb{N}$-shift?*

The main new technical idea of this paper is a modification to the standard conveyor belt trick by allowing "floating boundaries" around the support of a configuration: the 0s around the support are replaced with $>$s on the left, and $<$s on the right.

This contruction is explained in the proof of Theorem 3 in general, and Lemma 3 shows how to get wreath products out of it. In Example 2 we give a more explicit construction in the specific situation of embedding $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$. Section 6 gives a simpler construction for $\mathbb{Z} \wr \mathbb{Z}$, and illustrates why naive iteration of the construction in Lemma 3 does not lead to iterated permutational wreath products.

Theorem 3 and Lemma 3 specifically say that we get wreath products $A \wr H$ from "pointy" actions of groups H, which simply means there is a free orbit on a configuration with finite 0-support. Under a weaker condition on the action on finite points, we also obtain some permutational wreath products (Lemma 2).

The standard constructions of lamplighter-type groups (with finite base) and free groups are already pointy. We are in fact not aware of many other interesting pointy actions. One could answer Question 2 in the positive by finding pointy actions for the top groups. (In which case one would even obtain embeddings for the variants with $\mathbb{Z}$ as the base group.)

Question 4. *Do the groups $\mathbb{Z}_2 \wr \mathbb{Z}^2$, $\mathbb{Z} \wr \mathbb{Z}$, or $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ admit pointy actions, i.e. representations by elements of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ so that some configuration of finite 0-support has free orbit?*

We cannot even rule out that $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ itself admits an abstract pointy action.

In Section 6.3, we show that the simpler construction of Section 6 gives rise to some Neumann groups (from [17]) in $\mathcal{G}$, specifically the ones coming from eventually periodic sequences.

Question 5. *Are there Neumann groups corresponding to aperiodic sequences in $\mathcal{G}$?*

2 Definitions

The identity element of a group G is e_G; $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$. In a wreath product $K \wr H$ we usually refer to K as the *base* and H as the *top* group; such a group is a semidirect product $K^H \rtimes H$ where H acts by the regular action. The free group on n generators is F_n. If a group G acts on a set X, an *orbit* is $Gx = \{gx \mid g \in G\}$ for some $x \in X$, and a *free orbit* is Gx such that $g \mapsto gx$ is injective.

Throughout, Σ is a finite discrete set called the *alphabet*. The set $\Sigma^{\mathbb{Z}}$ is called the *full shift*, and it is a compact dynamical system under the *shift*, which is the homeomorphism $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$. We write $\mathbb{Z}_n$ for the cyclic group with n elements. The elements of $\Sigma^{\mathbb{Z}}$ are called *points* or *configurations*.

The set Σ^+ denotes all finite words over the alphabet Σ, i.e. the free semigroup on generators Σ with concatenation as multiplication. Write Σ^* for the corresponding free monoid adding the empty word. We write concatenation of $u, v \in \Sigma^*$ as $u \cdot v$ or just uv. We also work with left-infinite words $x \in \Sigma^{-\mathbb{N}}$ and right-infinite words $\Sigma^{\mathbb{N}}$, and can concatenate these together in various ways, with obvious interpretations, e.g. words from $x \in \Sigma^{-\mathbb{N}}, y \in \Sigma^{\mathbb{N}}$ and $w \in \Sigma^*$ join together into a point $x.wy \in \Sigma^{\mathbb{Z}}$, where . is used used denote where the new origin is.

A word or configuration over alphabet Σ^k can be interpreted as k *tracks* of configurations over Σ. This simply refers to coordinatewise projection. Beware that for $k = 2$, we often instead want to interpret a word $w \in (\Sigma^2)^*$ as a *conveyor belt*, as in Figure 1, in which case the second track is read in reverse. We will use the term "conveyor belt" and write our formulas to clarify when this is done.

Sometimes we will define a notion that is really about positions in a configuration, such as defining a position $i \in \mathbb{Z}$ to be *good* (in a configuration or word x) if it is contained in an interval $[a, b]$ such that the word $x_{[a,b]}$ has some property. We will in such discussions instead speak of particular symbols or subwords being good, and say things like "the subword v in uvw is good", meaning really that the interval where we explicitly write the subword v in uvw is good in whatever configuration, or *context*, uvw is being taken from. This should not cause confusion.

The finite set $\Sigma^{\mathbb{Z}_n}$ can be thought of as a $\mathbb{Z}$ or a $\mathbb{Z}_n$-system in an obvious way, with dynamics given by the map σ defined by the same formula. For $n \in \mathbb{Z}_+$, the subset of $\Sigma^{\mathbb{Z}}$ of points x satisfying $\sigma^n(x) = x$ is in natural correspondence with $\Sigma^{\mathbb{Z}_n}$. Such points are called *n-periodic*. We write elements of $\Sigma^{\mathbb{Z}_n}$ the same way as we write words in Σ^n (i.e. words Σ^n of length n over alphabet Σ).

The *automorphism group of the full shift* $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ consists of shift-commuting homeomorphisms $f : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$, which we call *automorphisms*. They are precisely the *reversible cellular automata* i.e. the bijective maps which admit a finite *neighborhood* $N \subset \mathbb{Z}$ and a *local rule* $f_{\mathrm{loc}} : \Sigma^N \to \Sigma$ such that $f(x)_i = f_{\mathrm{loc}}(x_{i+N})$. The group $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ is countable (due to local rules), residually finite (since periodic points are dense in $\Sigma^{\mathbb{Z}}$ and closed under $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$), and is not finitely generated [4].

Our alphabet Σ typically contains a special symbol 0 called *zero*, which can be taken to be part of the structure. The *nonzero* symbols are the ones that are not zero. Write $\mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$ for the point-stabilizer of $0^{\mathbb{Z}}$ in $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$. It is a finite-index subgroup, and contains an embedded copy of $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$ [25]. A point $x \in \Sigma^{\mathbb{Z}}$ is *0-finite* if $S = \{i \in \mathbb{Z} \mid x_i \neq 0\}$ is finite. Then S is the *0-support* of x.

3 The main technical result

Definition 1. *Let $\Sigma \ni 0$ be a finite alphabet. A set of finite words $u_i \in \Sigma^+$ is n_0-safe if: the words u_i begin and end with nonzero symbols, are distinct (in particular these two imply that $\ldots 000u_i000 \ldots$ have disjoint shift orbits), $\max |u_i| \leq n_0$ and we can uniquely deduce u_i and j from $\sigma^j(u_i 0^{n-|u_i|})$ for any $n \geq n_0$.*

Here, $u_i 0^{n-|u_i|}$ is seen as an element of $\Sigma^{\mathbb{Z}_n}$, and σ is the cyclic shift. For example, for $u_1 = 101$, the set $\{u_1\}$ is not 4-safe, since $1010 = u_i 0 = \sigma^2(u_i 0)$, but it is 5-safe. In general, if u_i are distinct words that begin and end with nonzero symbols, then $U = \{u_1, \ldots, u_n\}$ is n_0-safe for any $n_0 \geq 2 \max_i |u_i| - 1$, in particular if all the u_i have just one letter, $\{u_1, \ldots, u_n\}$ is 1-safe.

Definition 2. *Let $X = \bigcup_{n \in \mathbb{Z}_+} \Sigma^{\mathbb{Z}_n}$. Define G to be the smallest group of permutations on X satisfying the following:*

- *G contains all (permutations corresponding to) elements of $\mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$, applied to elements of $\Sigma^{\mathbb{Z}_n}$ interpreted as n-periodic configurations, and*
- *for any n_0, any n_0-safe finite set $\{u_1, \ldots, u_k\} \subset \Sigma^+$, permutation $\pi \in S_k$ and $n_1, \ldots, n_k \in \mathbb{Z}$, G contains the map that acts trivially on $\Sigma^{\mathbb{Z}_n}$ with $n < n_0$, and on $\Sigma^{\mathbb{Z}_n}$ with $n \geq n_0$ acts by*

$$\sigma^j(u_i 0^{n-|u_i|}) \mapsto \sigma^{j+n_i}(u_{\pi(i)} 0^{n-|u_{\pi(i)}|}).$$

  for all j (and trivially on words not of this form).

Note that G fixes the sets $\Sigma^{\mathbb{Z}_n}$, and so is obviously residually finite. We often refer to the elements of $\Sigma^{\mathbb{Z}_n}$ as *tapes*. Elements of the form described in the second item are called *AFO* (from "actions on finitely many orbits"). We refer to the numbers n_i as *offsets*.

Example 1: For example, pick $\Sigma = \{0,1,2\}, u_1 = 1, u_2 = 2, n_0 = 1, \pi = (1\ 2), n_1 = 1, n_2 = -1$. Then the AFO f corresponding to this data maps

$$f(1) = 2, f(20) = 01, f(12) = 12, f(00010) = 00002, f(001000100) = 001000100.$$

Usually, we do not give the data for AFOs explicitly, but simply explain in words which configurations are shifted or modified, and how. ○

AFOs are a priori quite different from automorphisms, as they can in a sense read arbitrarily many symbols of a configuration in order to decide whether or not to modify or shift it, while automorphisms admit local rules. More generally, it may seem that compactness is a fundamental obstacle to performing such feats. The trick around this is simple: we add new symbols that allow us to explicitly mark infinite all-zero tails.

Lemma 1. *The embedding* $\mathrm{Aut}_0(\Sigma^{\mathbb{Z}}) \to G$ *is split.*

Proof. Recall that this means there is a retraction, i.e. a homomorphism from G to $\mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$ that maps trivially on the subgroup $\mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$. Consider a product g of elements of H and AFOs. If we pick a large n and a uniformly random element of Σ^n, with high probability none of the AFOs actually act (since the support of an AFO is of polynomial size).

In particular, for large enough n, g eventually acts by a fixed automorphism in H on more than half of the inputs (which is of course precisely the element g' obtained from the product representation of g by dropping the AFOs). It is also clear that for any other element $h \in G$, eventually it acts differently from g' on almost all elements of Σ^n (since with high probability we see all possible contents of the neighborhoods of both automorphisms). The retraction is then map that takes g to the unique g' by which it eventually acts with probability greater than $1/2$. □

Theorem 3. $G \in \mathcal{G}$.

Proof. We first perform the construction with $X = \bigcup_{n \in 2\mathbb{Z}_+} \Sigma^{\mathbb{Z}_n}$ (so we embed the group described exactly like G above, but with X only containing even-length tapes), and explain how to modify it for the union over $\mathbb{Z}_+$.

As our first step, we construct a new action of H (again this is the point stabilizer of $0^{\mathbb{Z}}$ in $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$) on a different full shift. As usual, we will use the idea of “conveyor belts” [20] meaning we simulate the action of H on paths that are wrapped as in Figure 1, the point being that two pieces of tape with different orientations can be glued together into a continuous tape if we discover some kind of a problem.

The new idea is to use *conveyor belts with floating boundaries.* This means that we will allow (though cannot quite force) the tape to contain explicit markings that tell us where the 0-support of a configuration lies on the conveyor belt. Thus, in some situations we are able to locally detect that we are dealing with precisely a particular configuration $\sigma^j(u0^{n-|u|})$ and not just a very good approximation of it.

We use the alphabet $\Gamma = \Sigma^2 \cup \{<, >\}$. Write $\mathbf{0} = (0,0), B = \Sigma^2, C = B \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is thought of as the zero symbol of Γ. We will refer to a symbol $>$ (resp. $<$) as a *wall* if the symbol to its left (resp. right) is **not** $>$ (resp. $<$). A *good subword* is any word in

$$\{>>, >C, BB, C<, <<, ><\}.$$

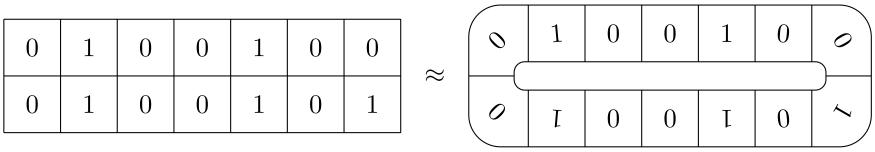

Figure 1: A word of length n over Σ^2 with $\Sigma = \{0,1\}$ can be interpreted as a conveyor belt containing a word of length $2n$, by concatenating the word on the top track to the reversal of the word on the bottom track.

Here and below we abuse notation, and by BB denote any word in $BB = \{bb' \mid b, b' \in B\}$; and similarly for $>C$ and $C<$. Other words of length 2 are called *bad*; explicitly these are

$$\{<>, B>, <B, >\mathbf{0}, \mathbf{0}<\}.$$

A symbol that is only part of good words (i.e. the two words of length two that contain it are both good) is called *good*. If a symbol is part of a bad word ab, and it is not a wall then it is called *error*.

Let $f \in H$. We will describe the action of f on doubly transitive points[1] in such a way that it simply simulates the natural action of H on encoded configurations. This action works as follows. No bad symbol (i.e. error or wall) is modified. Good symbols are part of maximal *good runs*, i.e. words v of the form $>^*u<^*$ where $u \in B^*$, and either u is the empty word, or the first and last symbols in u are nonzero. Concretely, v is in $>^*CB^*C<^* \cup >^*C<^* \cup >^*<^*$.

We deal with such words differently depending on whether the symbols to the left and right are wall or error. Consider first the case where the symbols on the left and right are walls, i.e. the context is $a> \cdot >^m u <^n \cdot <b$ where $a \neq >$, $b \neq <$.

Our action of H will rewrite the subword $>^m u <^n$. If $|u| = 0$, the action of f is trivial. Otherwise, interpret the symbols $>$ and $<$ as $\mathbf{0}$s, and then read the resulting word $w = \mathbf{0}^m u \mathbf{0}^n$ as a pair of words $s, t \in \Sigma^*$ by separately reading the word on the top track and bottom track, i.e. $s_i = (w_i)_1, t_i = (w_i)_2$.

Now apply f to the periodic point $(st^R)^{\mathbb{Z}}$ to obtain a point $(s't'^R)^{\mathbb{Z}}$ with $|s'| = |s|, |t'| = |t|$. Then reverse the process by writing $w' = \begin{smallmatrix} s' \\ t' \end{smallmatrix}$ (i.e. the word defined by $w'_i = (s'_i, t'_i)$) as $w' = \mathbf{0}^{m'} u' \mathbf{0}^{n'}$ with m', n' maximal. Observe that the left- and rightmost symbols in u' are nonzero, and u' is not the empty word (since automorphisms cannot map nonzero points to the zero point), so this representation of the word is unique. Finally write $v' = >^{m'} u' <^{n'}$ back onto the tape. Note that the maximal good run remains the same, as the surrounding words $a>$ and $<b$ will not be modified (as all of their symbols are bad).

Next, suppose the symbol to the left of v is an error, and the symbol to its right is a wall. The context is then $ab \cdot >^m u <^n \cdot <c$, where $c \neq <$ and ab is one of the cases $>\mathbf{0}$, $<B$, $\mathbf{0}<$. In each case, we must have $m = 0$, as otherwise the leftmost $>$ would in fact be a wall, i.e. our situation is actually $ab \cdot u <^n \cdot <c$ with $u \in B^*$, and the last symbol of u is nonzero, unless u is the empty word.

If $ab = \mathbf{0}<$, then u must in fact be the empty word. If $ab = >\mathbf{0}$, then u

[1] Doubly transitive points are dense, so it suffices to describe the action on these points.

cannot be the empty word, as then its first symbol $<$ of $u{<}^n$ is not good. If $ab = {<}\mathbf{0}$, the same is true. If $ab = {<}C$, then u can be empty or nonempty.

Now, if u is the empty word, we do not modify the word $u{<}^n$. Otherwise, we work exactly as above: we replace $<$-symbols by $\mathbf{0}$s (this time, there are no $>$-symbols); then we applying f to the corresponding conveyor belt, then we rewrite the maximal $\mathbf{0}$-suffix as ${<}^*$. The difference to the previous case is that we leave the maximal $\mathbf{0}$-prefix intact and write any initial zeroes as zeroes back on the tape.

Observe that this means we write back a word ending in a nonzero symbol, followed by $<$ symbols (again because no nonzero point maps to the all-zero point under f), in particular we always write a symbol from B to the right of ab, so all the symbols written on top of the good run will be good, since the surrounding bad symbols ab and ${<}c$ are again not modified.

If there is an error on the right, and a wall on the left, then we work as above, up to obvious left-right-symmetry. If there is an error on both sides, then the situation is $ab \cdot u \cdot cd$ where $u \in B^*$. We work exactly as in the previous cases, but rewrite neither the prefix nor the suffix of $\mathbf{0}$s into $>$ or $<$ in the last step, and just write the f-image back on the tape as such.

All in all, our description leads to an action of H, by automorphisms. Bijectivity is clear, since we are simply applying bijective transformations to simulated configurations through a natural interpretation that stays consistent between applications, so f and f^{-1} will be mapped to their inverses under this construction. Shift-commutation is obvious, and for continuity one simply checks that to determine the new symbol at the origin one only has to read $O(1)$ elements to determine the local conveyor belt structure, and apply f to the simulated configuration; determining the final $>$ and $<$-rewrites is equally local, since f fixes the point $0^{\mathbb{Z}}$.

As for AFOs, in situations where the conveyor belt has walls on each side (the first case of the four we considered), we can locally detect the entire configuration. Thus, we are able to shift around and permute any finite set of configurations.

Finally, to replace $2\mathbb{Z}_+$ by $\mathbb{Z}_+$, we can simply add a preprocessing step where we replace the neighborhoods $N \subset \mathbb{Z}$ of elements of H by $2N$, so that H effectively acts on two independent copies of a full shift. Similarly, we can replace all the words u_i by words u_i' where 0s have been inserted between any two symbols of u_i and at the end, and double the values n_0 and $n_1, \ldots, n_k$.

The subgroup obtained from the above construction is isomorphic to G: consider the new action on good runs of length n, i.e. a conveyor belts simulating periodic configurations of length $2n$, possibly with $>$- and $<$-affixes. The G-orbits where both the odd and even cells contain nonzero symbols will simulate natural H-orbits since we will never detect the shift-orbits of any $u_i' 0^{2n-|u_i'|}$, so by the previous lemma we get only the relations in G. If the odd positions contain only zeroes, then they continue to do so as elements of H or AFOs are applied, and we are precisely simulating the action of G on a single tape of size n. $\square$

For the construction above it suffices that the automorphism be "0-to-0" in the terminology of [27] i.e. the only preimage of $0^{\mathbb{Z}}$ should be $0^{\mathbb{Z}}$. A general Python implementation of this construction can be found on [26].

4 Pointy actions

Definition 3. *Let $H \leq \mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$. We say H is* pointy *if some 0-finite configuration called the* special point $x_0 \in \Sigma^{\mathbb{Z}}$ *has free orbit under H. We say H is* weakly pointy with special point $x_0 = \ldots 000u000\ldots$ *if the point stabilizer of x_0 is contained in the point stabilizer of $u0^{n-|u|}$ for large enough n (for the natural action of H on $\Sigma^{\mathbb{Z}_n}$).*

Note that a weakly pointy action is indeed a lot weaker than a pointy one, for example one can check that if H is finitely generated and fixes the point x_0, then its action is weakly pointy for this special point.

For a group $H \leq \mathrm{Aut}(\Sigma^{\mathbb{Z}})$ and any point x_0, write $A \wr_{x_0} H$ for the permutational wreath product $A \wr_{\Omega} H$ where $\Omega = Hx_0$ and H acts naturally on Ω. We use a similar convention for subsets of $\Sigma^{\mathbb{Z}_n}$.

Lemma 2. *Suppose $H \leq \mathrm{Aut}_0(\Sigma^{\mathbb{Z}})$ is weakly pointy with special point x_0. Then $A \wr_{x_0} H \in \mathcal{G}$ for all finitely generated abelian groups A.*

Proof. It suffices to show $A \wr_{x_0} H \in \mathcal{G}$ for finite abelian A and infinite cyclic A, since $(N \times K) \wr_{\Omega} H \leq (N \wr_{\Omega} H) \times (K \wr_{\Omega} H)$. We start with the case of a finite A, whose group operations we will write additively.

Suppose H acts weakly pointily with special point $x_0 = \ldots 000.u000\ldots$. Take alphabet $\Sigma \times A$ with zero element $(0, 0_A)$, and have H act on the first track, ignoring the second. Now consider the subgroup of G generated by H, and the AFOs that take $(u0^{n-|u|}, a0^{n-1})$ to $(u0^{n-|u|}, (a+a')0^{n-1})$ for various a', for all $n \geq n_0$ (but do nothing else).

We show that these maps form a copy of $A \wr_{x_0} H$ inside G. Observe that this abstract group is generated by H and generators of A applied "at" (approximations of) x_0, and the conjugation action of H is to move the points by its natural action (carrying with them the element of A that has been applied on each point).

First, consider G as embedded in the automorphism group of a full shift according to the proof of Theorem 3. Consider tapes where x_0 appears in the form $>^{-\mathbb{N}}((u, a0^{|u|-1}), 0^{|u|})<^{\mathbb{N}}$. Clearly on such points we are simulating in a concrete way this abstract action of the group: H literally acts by its natural action on the top track, and the generators for the bottom A (the AFOs) add a' to the unique symbol on the bottom track, if the top track contains precisely the configuration x_0. Thus, the group we have constructed inside G admits an epimorphism onto $A \wr_{x_0} H$.

Then again consider G abstractly. By a similar analysis as in the previous paragraph, the action of the subgroup of G we constructed, on the set $\Sigma^{\mathbb{Z}_n}$, is just the natural action of $A \wr_{u0^{n-|u|}} H$. By the assumption on stabilizers, the action $H \curvearrowright O_H(u0^{n-|u|})$ is a factor of $H \curvearrowright O_H(x_0)$ (by mapping $h \cdot x_0 \mapsto h \cdot u0^{n-|u|}$. Thus $A \wr_{u0^{n-|u|}} H$ is a factor of $A \wr_{x_0} H$ (since A is abelian).

For $\mathbb{Z} \wr_{x_0} H$, we do the same, but now instead of acting on $\Sigma \times A$ we act on $\Sigma \times \{0, 1\}$. We use the natural action of H on the first track, and then in G we consider the subgroup containing H and the AFO that shifts points from $O((u0^{n-|u|}, 10^{n-1}))$ by offset 1, and fixing other points.

Again, on infinite conveyor belts (in the sense of the embedding of G into the automorphism group of a full shift) we have the natural action of $\mathbb{Z} \wr_{x_0} H$: when shifting on the base group $\mathbb{Z}$ we shift a unique bit on the second track.

As a side-effect we shift the entire configuration, but the action of H commutes with the shift so we can ignore this and only keep track of the movement on the second track: the total shift on the first track is invisible to the action of H, and its total shift exactly copies the total shift of the second track does.

On finite conveyor belts, by a similar analysis we have an action of the group $\mathbb{Z}_n \wr_{u0^{n-|u|}} H$, which is a factor of $\mathbb{Z} \wr_{x_0} H$ by the assumption on stabilizers. $\square$

Lemma 3. *Suppose $G \in \mathcal{G}$ admits a pointy action. Then $A \wr G \in \mathcal{G}$ for all finitely-generated abelian groups A.*

Proof. A pointy action is weakly pointy, and we have $A \wr_{x_0} G \in \mathcal{G}$. Since the orbit is free, this is simply the group $A \wr G$. $\square$

5 Examples of pointy actions

Lemma 4. *The set of groups admitting pointy actions is closed under direct products, subgroups, and commensurability.*

Proof. For direct products, we use different tracks, one for each group, and the action $(g, h)(x, y) = (gx, hy)$. As the special point x_0 we use the pair of special points of the two actions, one on each track.

For subgroups, obviously for any pointy action, the subaction of any subgroup is also pointy.

For commensurability, since we have closure under subgroups, it suffices to show that if G is of finite index in H and G admits a pointy action, then so does H. The Krasner–Kaloujnine universal embedding theorem states that H embeds in $G \wr S_n$ for some n (S_n is the symmetric group on n points). Since we have direct products, we have a pointy action of G^n, and we can combine this with the action of swapping the contents of various tracks to implement the action of S_n (it is easy to keep the action free). $\square$

Lemma 5. *Every countable free group admits a pointy action.*

Proof. It suffices to consider the free group on two generators. Free nonamenable groups on finitely many generators are pairwise commensurable, and by a theorem of Nielsen [14, Theorem 2] they are the same as finite free products of finite groups up to commensurability, so it suffices to consider finite free products of finite groups.

Alperin constructs an action for such groups in [1]. His proof in fact shows that this action is pointy. Specifically, the action on $\ldots 111{*}111 \ldots$ is free (i.e. we take the identity element as the zero symbol). $\square$

Lemma 6. *Let A be a finite abelian group. Then the group $A \wr \mathbb{Z}$ admits a pointy action.*

Proof. The action constructed for these groups in [23] (explained also in the example below) is easily seen to be pointy. The orbit of the configuration $\begin{matrix} \ldots 0001000 \ldots \\ \ldots 0000000 \ldots \end{matrix}$ is free. $\square$

Example 2: We illustrate the embedding of $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ obtained by combining Lemma 3 with the previous lemma. We will refer to $\mathbb{Z}_2 \wr \mathbb{Z}$ as the *top lamplighter*, and the $\mathbb{Z}$ on the left as the *bottom* $\mathbb{Z}$. The generators of the top lamplighter will be called L, R (left and right; the generators corresponding to a symmetric generating set for the $\mathbb{Z}$ quotient) and F (flip; the generator of $\mathbb{Z}_2$ at $0_{\mathbb{Z}}$), and U, D (up and down) are the generators for the bottom $\mathbb{Z}$ at the identity of $\mathbb{Z}_2 \wr \mathbb{Z}$.

We use the usual action of $\mathbb{Z}_2 \wr \mathbb{Z}$ on $(\{0,1\}^2)^{\mathbb{Z}}$ (see e.g. [23]): the generator of $\mathbb{Z}$ shifts the first track, and the generator of $\mathbb{Z}_2$ (at $0_{\mathbb{Z}}$) sums the first track to the second.

The way we implement $\mathbb{Z} \wr (\mathbb{Z}_2 \wr \mathbb{Z})$ in G is now that we take the above copy of $\mathbb{Z}_2 \wr \mathbb{Z}$ in H. Then we modify its action according to the proof of Theorem 3 so that it acts on conveyor belts with floating boundaries. The generator U for the bottom $\mathbb{Z}$ (at the origin of top lamplighter) is implemented by shifting the word $(1, 0, 1)$ (seen as a word of length 1, over the alphabet $\{0, 1\}^3$) by 1, i.e. the AFO data is $u_1 = (1, 0, 1)$, $n_1 = 1$, $n_0 = 1$; D is its inverse.

We illustrate the action of this group in Figure 2, by computing a spacetime diagram for the element $(FL)^3 \cdot ULUFRD^4LFR$, in the sense of showing how partial applications of this product modify a particular configuration (which we have chosen so that it illustrates some of the relevant phenomena). The symbols $>, <$ are directly written as themselves, while we write an element of $(\{0,1\}^3)^2$ as a stack of three colored boxes, the top three boxes corresponding to the first $\{0,1\}^3$ (the top track) and the bottom three boxes corresponding to the latter $\{0,1\}^3$ (the bottom track). The fill color white corresponds to 0, and all other colors correspond to 1.

There are three good runs visible. On the leftmost the top lamplighter configuration corresponds to the identity, so $ULUFRD^4LFR$ adds 1 to the bottom copy of $\mathbb{Z}$, and then $(FL)^3$ flips a few more bits on the top lamplighter side. In the middle conveyor belt, the top lamplighter is in state LFR, so the initial $ULUFRD^4LFR$ instead subtracts 4 from the bottom copy of $\mathbb{Z}$. Note that now $(FL)^3$ moves to the right, being on the bottom tape.

On the third tape we just see the action of $(FR)^3(LFR)^2$ of the top lamplighter for three reasons: there is an error on the right (so the right end does not float), there are two top lamplighters (i.e. 1s on the simulated top track), and there are two 1s on the simulated bottom track (corresponding to the bottom $\mathbb{Z}$). Note that between the second and third good run there is also a zone with only errors, which is never modified. ○

The Python script at [26] generates tikz code for the figure in the example.

The following is immediate from the closure properties and constructions in this section. Combining it with Lemma 3 immediately yields the results mentioned in the introduction.

Lemma 7. *If B is a finite abelian group, C is a finitely-generated free group, and $n \in \mathbb{N}$, then $A \wr ((B \wr \mathbb{Z})^n \times C^n)$ is in $\mathcal{G}$.*

6 A simpler construction that gives $\mathbb{Z} \wr \mathbb{Z}$ and some other groups

We give here a simpler variant of the construction from Theorem 3, which does nothing more than marks individual positions on conveyor belts (and has no

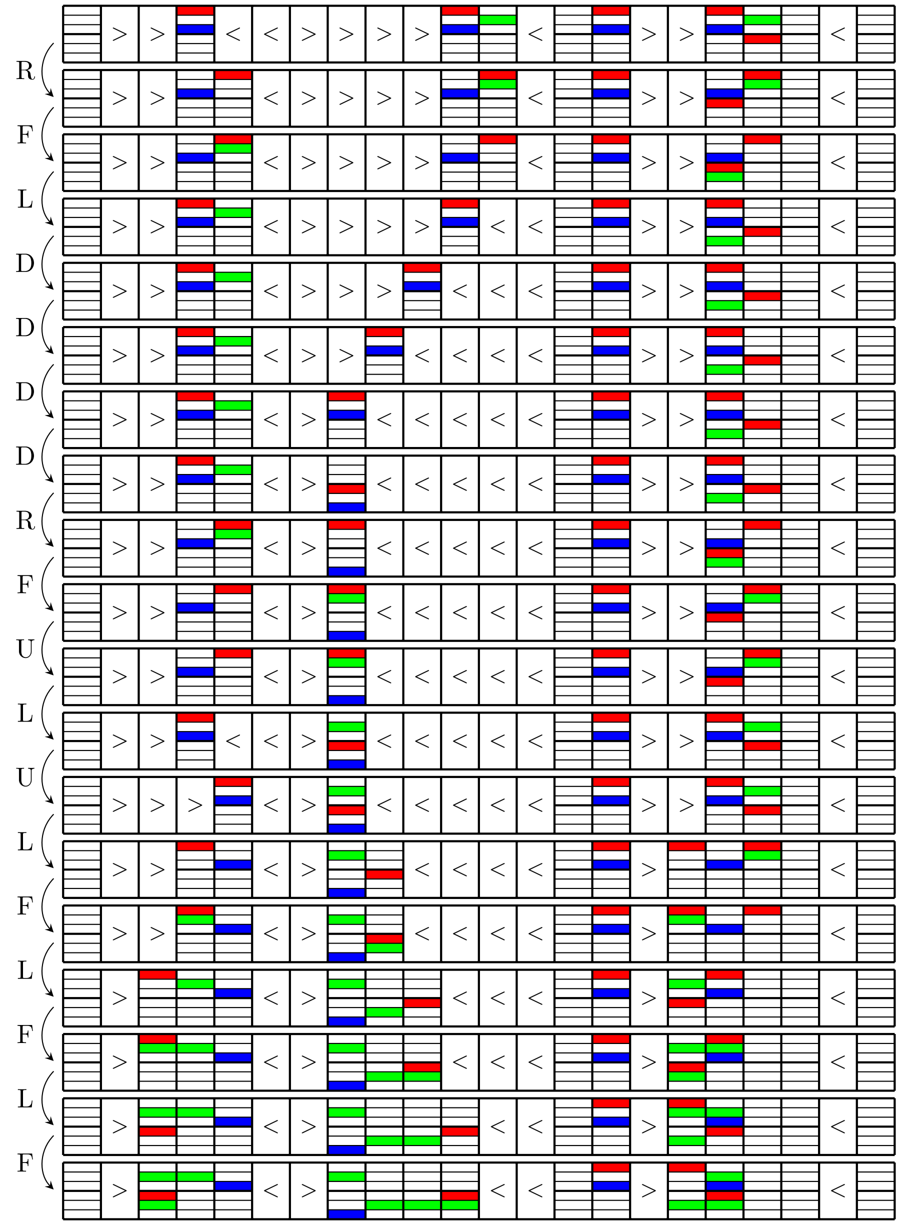

Figure 2: A spacetime diagram for the element $(FL)^3 \cdot ULUFRD^4LFR$.

explicit conveyor belts). In Section 6.2, we give an algebraic description of some solvable groups obtained from, in a sense, iterating Lemma 2 for this construction, illustrating how the final group differs from a standard iterated permutational wreath product. In Section 6.3, out of general interest, we show how this construction gives rise to some Neumann groups in $\mathrm{Aut}(\Sigma^{\mathbb{Z}})$.

6.1 The simplified construction

The construction roughly follows that of Theorem 3, and the reader should be familiar with the proof. We consider the alphabet $\{1,2,>,<\}^k$, $C = \{1,2\}$. In a point of $\{1,2,>,<\}^{\mathbb{Z}}$, we designate $\{>C, C<\}$ as *good words*, while the remaining words $\{CC, <C, C>, ><\}$ are *bad*. In a point over the alphabet $\{1,2,>,<\}^k$, a coordinate is *good* if it is good on all the n tracks individually. Now, each doubly transitive configuration splits into finite maximal good runs, meaning maximal runs where on each track we only have letters which are not contained in a bad word. A good run over the alphabet $\{1,2,>,<\}^k$ will give a good run on each single track, and these runs are of the form $>^*C<^*$, $>^+$, $<^+$, with the left- and rightmost letters extending to a good word on each side. Of course, bad words on any one track will also cut good runs on all other tracks so that they may no longer be maximal on that track.

It is clear that on good words of length n on an individual track, we have an action of $\mathbb{Z}_{2n}$ which moves 1-symbols to the left, and 2-symbols to the right, except at the left end it flips 1 to 2 and at the right end it flips 2 to 1. Again we can think of this as a conveyor belt, with a single 1 moving to the left on the "top track" and 2 moving to the left on the "bottom track". This action does not change the set of good subwords on the track it modifies, so it also does not change the set of good positions of the composite configuration over $\{1,2,>,<\}^k$.

More generally, if we only make modifications to good runs $>^m a <^n$ by changing the values of m and n, and possibly changing a to $3-a$ (even depending arbitrarily on the other tracks!), we will not modify the global set of areas where good runs appear. From this one gets a large variety of different behaviors, some of which are explored in the following sections.

6.2 Some solvable groups obtained

Let us now look at what kind of group we get if each head only affects heads with higher index. This type of construction intuitively should give a solvable group, since taking the commutator subgroup effectively eliminates one head at a time.

Definition 4. *Let $n \in \mathbb{N}$ and let $K_1^n = \mathbb{Z}/n\mathbb{Z}$ (i.e. $K_1^0 = \mathbb{Z}$, and $K_1^n = \mathbb{Z}_n$ for $n \geq 1$). Then K_1^n acts on $\mathbb{Z}/n\mathbb{Z}$ by the regular (translation) action. By induction, K_k^n acts on $(\mathbb{Z}/n\mathbb{Z})^k$, and we define $K_{k+1}^n = \mathbb{Z}/n\mathbb{Z} \wr_{(\mathbb{Z}/n\mathbb{Z})^k} K_k^n$, which acts naturally on $(\mathbb{Z}/n\mathbb{Z})^{k+1}$: the top group K_k^n acts on the first k coordinates by its natural action, and the generator for the base $\mathbb{Z}/n\mathbb{Z}$ (the delta at $e_{K_k^n}$) acts by the regular action on the last coordinate if and only if other coordinates are 0.*

There is a natural generating set for K_k^n, where the generator g_i adds 1 to the ith coordinate if all smaller coordinates $\{1,2,\ldots,i-1\}$ contain 0 (modulo

n, when $n \geq 1$). Using these generators, we can consider K_k^n as elements of the space of marked groups on k generators [8]. The following is easy to show.

Lemma 8. *$K_k = \lim_n K_k^n$ in the space of marked groups.*

Definition 5. *Define $\hat{K}_k$ as the group generated by k elements $g_1, \ldots, g_k$ with relations precisely the relations that hold in all of the K_k.*

Theorem 4. *For all k, $\hat{K}_k \in \mathcal{G}$.*

We see below that $\hat{K}_k = \mathbb{Z} \wr \mathbb{Z}$, so this gives a simpler construction for this group in $\mathcal{G}$.

Proof. We use the above construction, on alphabet $\{1, 2, >, <\}^k$. As in the description above, we interpret good runs, which are either degenerate, or contain a unique *head* which is an element of $\{1, 2\}$. To a head we associate its *belt position* by $p(0^m 1 0^n) = m$, $p(0^m 2 0^n) = (m + n + 1) + n$.

The automorphism corresponding to g_i adds 2 to the belt positions on tracks $\{1, \ldots, i\}$ if and only if there are indeed heads on all these tracks, and their belt positions are equal. Concretely, all of the tracks should have a 1-symbol in the same position, or all should have a 2-symbol in the same position. (We move the heads' belt positions 2 steps at a time simply so that we effectively get also tapes of odd length.)

It is immediate that on a single good run, where all heads are present and in belt positions with the same parity, this mimics the action above. To conjugate the actions when the parities are even, the tuple of belt positions $(2h_1, \ldots, 2h_k)$ is mapped to $(h_1 - h_2, h_2 - h_3, \ldots, h_{k-1} - h_k, h_k)$. As in the main construction (for wreath products with base $\mathbb{Z}$), the point is that our automorphisms will also move the first $i - 1$ heads when moving the ith, but in any case once the ith head returns back, this shift has been undone.

If ℓ is minimal such that a head is missing from the ℓth track in some good run (i.e. that tape is degenerate) or the belt position of the head has a different parity than head on the previous track, then on that particular good run we simply mimic the quotient action of $K_{\ell-1}^n$. $\square$

The following proposition illustrates (for this specific example) that iterating the main construction does not lead to an iterated permutational wreath product (K_k), but instead to a product of finite iterated permutational wreath products ($\hat{K}_k$), and these can be different.

Proposition 1. *The map $g_i \mapsto g_i$ gives a homomorphism from $\hat{K}_k \to K_k$. The group $\hat{K}_k$ is equal to K_k if $k \leq 2$. If $k \geq 3$, then K_k is torsion-free, but $\hat{K}_k$ is not. All of the groups K_k, $\hat{K}_k$ are solvable.*

Proof. The first statement is clear from $K_k = \lim_n K_k^n$. For the second, it is easy to verify that $\hat{K}_1 = K_1 = \mathbb{Z}$ and $\hat{K}_2 = K_2 = \mathbb{Z} \wr \mathbb{Z}$. It is also easy to see that the groups K_k are all torsion-free and $\hat{K}_{k+1}$ contains a copy of $\hat{K}_k$, so for the third statement, it now suffices to show that $\hat{K}_3$ is not torsion-free.

Let $n \geq 2$. We observe that $h = g_2^{g_1^n}$ stabilizes 0^2 in K_2, but does not stabilize 0^2 in the action of K_2^n. This means $g_3^h g_3^{-1}$ is trivial in K_3 but not in K_3^n. Since K_3 is the limit of the groups K_3^ℓ, $g_3^h g_3^{-1}$ is trivial in K_3^ℓ for sufficiently large ℓ. Thus its order is bounded over the finitely many finite groups K_3^ℓ where it acts nontrivially, and therefore its order is finite in $\hat{K}_3$.

For the last statement, the group is solvable of derived length at most k by definition as an iterated permutational wreath product (one can check that indeed it is precisely k). Consider now for $N \subset \mathbb{N}$ the group K_k^N where we take generators g_i under precisely the relations that hold in all of the K_k^n. This is just the subgroup of $\prod_{n \in N} K_k^n$ generated by the k many diagonal elements $(g_i, g_i, g_i \ldots)$. Now we observe that the entire group $\prod_{n \in N} K_k^n$ is solvable of derived length at most k, as a product of groups of derived length at most k. For this, recall that solvable groups of degree k form a variety, or concretely observe that $[\prod_i G_i, \prod_i G_i] \leq \prod_i [G_i, G_i]$ for any groups G_i by a direct computation. (Again one can check that the derived length of $\hat{K}_k$ is precisely k.) $\square$

6.3 Neumann groups

Next, let us look at what we get when one has a single head.

Definition 6. *Let $3 \leq n_1 < n_2 < \ldots$ be positive numbers, and let $u_{i,j}$ be distinct elements for $1 \leq j \leq n_i$. The* Neumann group *corresponding to the sequence (n_i) data is the subgroup of the product of alternating groups $\prod_i \mathrm{Alt}(\{u_{i,j} \mid j \leq n_i\})$ generated by the permutations $a = \prod_i (u_{i,1};\ u_{i,2};\ u_{i,3})$ and $b = \prod_i (u_{i,1};\ u_{i,2};\ \ldots;\ u_{i,n_i})$.*

Neumann shows in [17] that two such groups are isomorphic if and only if the defining sequences are the same. Since he assumes the numbers are odd, and the even case is the most natural one in our present context, we recall the proof.

Lemma 9. *Let G be the Neumann group coming from a sequence $5 \leq n_1, n_2, \ldots$. Then the isomorphism type of G determines the sequence $(n_i)_i$.*

(We start from 5 for simplicity's sake.)

Proof. The element $[a, a^{b^{n+3}}]$ is nontrivial in A_{n+5} (where $a = (1;\ 2;\ 3)$ and $b = (1;\ 2;\ \ldots;\ n)$), and trivial in A_{n+m} for $m \geq 6$. By induction on i, it is then easy to show that the natural A_{n_i} in G (other A_n acting trivially) is a subgroup of G, as once we have built $A_{n_{i-1}}$, the normal closure of $[a, a^{b^{n_i - 2}}]$ is a subgroup of $\prod_{h \leq i} A_{n_i}$ surjecting to A_{n_i}, and we can cancel the smaller A_{n_h}. Furthermore, this A_{n_i} is clearly normal.

On the other hand, consider any finite normal subgroup of G. Its elements clearly have nontrivial projection to only finitely many of the $\mathrm{Alt}(\{u_{i,j} \mid j \leq n_i\})$. It is then a straightforward consequence of Goursat's lemma [7] that if such a group is simple, it is precisely one of the $\mathrm{Alt}(\{u_{i,j} \mid j \leq n_i\})$. All in all the n_i are determined by isomorphism types of finite simple normal subgroups of G. $\square$

Lemma 10. *Let $3 \leq n_1 < n_2 < \ldots$ be any eventually periodic sequence, meaning for some $p \geq 1$ we have*

$$\exists n_0 : \forall n \geq n_0 : n \in \{n_i \mid i \in \mathbb{Z}_+\} \iff n + p \in \{n_i \mid i \in \mathbb{Z}_+\}.$$

Then the corresponding Neumann group G is in $\mathcal{G}$.

Proof. First we consider the sequence $6 < 8 < 10 < \ldots$. We use the alphabet $\{>, <, 1, 2\}$ and the same rule as before the determine good runs. In good runs of length $n \leq 2$, we do nothing. For larger n, the automorphism corresponding to a rotates the belt positions (1; 2; 3) meaning at the beginning of each good run we perform the permutation (1<<; >1<; >>2). The automorphism corresponding to b is just the rotation of the head around the conveyor belt, as in the previous section. Obviously on finite good runs this is isomorphic to the natural action of the Neumann group.

For other sequences, it suffices to construct the natural action of the group for a single sequence $\ell, \ell + k, \ell + 2k, \ldots$, since we can then take the product action of Neumann groups corresponding to finitely many such sequences to get any Neumann group with an eventually periodic sequence.

Consider the alphabet $\{>, <, 1, 2\} \times \{1, 2, \ldots, k\}$. First, use the same rule as above to cut the sequence on the first track over $\{>, <, 1, 2\}$ into good runs. Then in each good run, on the second track over alphabet $\{1, 2, \ldots, k\}$, read maximal concatenations w^n of the word $w = 12 \cdots k$. Cut the good word by these runs, including a full copy of w on the left, and only the first ℓ letters in the rightmost good run. This forces good runs to be effectively of length $\ell \bmod k$. Now the generators with the same description as above (but with the new description of good runs) form the Neumann group. □

We note that the previous construction can also be performed in the topological full group $[\![\Sigma^{\mathbb{Z}}]\!]$ of $\Sigma^{\mathbb{Z}}$ for some alphabet Σ (and therefore any nontrivial alphabet [24]), by for a single arithmetic progression defining good runs to consist of maximal powers of the word $w = 12 \cdots (2n) \in \{1, \ldots, 2n\}^{2n}$, and using the origin of the configuration as the head position (using odd and even positions instead of the symbol $s \in \{1, 2\}$ to determine whether we are on the top or bottom track of track of the conveyor belt). To get multiple arithmetic progressions, one can simply use finitely many disjoint alphabets.

7 Lemmas used in the introduction

Lemma 11. *Let A, B be finitely generated groups, such that A is abelian and B has co-NP word problem. Then $A \wr B$ has co-NP word problem.*

Proof. Let $\pi : A \wr B \to B$ be the natural projection. Pick a generating set S for B. We exhibit polynomially checkable witnesses for the nontriviality of an element in the wreath product. Since B has co-NP word problem, for elements with nontrivial $\pi(g)$ we can use the certificates of B. Elements with trivial $\pi(g)$ are finite products of the form $\prod_i a_i^{b_i}$ where $a_i \in A, b_i \in S^*$. Writing $b_i \sim b_j$ for equality as elements of B, the nontriviality of the product means that for some b, the sum (in A) of elements a_i corresponding to $b_i \sim b$ is nonzero. A polynomial witness then consists of witnesses for all non-equalities between unequal b_i (using again the co-NPness of the word problem of B), or sufficiently many to conclude that the element is nontrivial. □

See [16] for more information on decision problems about wreath products.

The following lemma shows that our Theorem 2 does not provide examples of torsion-free groups in $\mathcal{G}$ which are not residually nilpotent.

Lemma 12. *If A is a torsion-free abelian group, C is a finitely-generated free group, and $n \in \mathbb{N}$, then $A \wr C^n$ is residually nilpotent.*

Proof. For any quotient $\pi : C^n \to H$, there is a natural quotient $\hat{\pi} : A \wr C^n \to A \wr H$ (since A is abelian) mapping as π on C^n and as identity on A. The group F_n is residually torsion-free nilpotent [15], so for any nontrivial element $g \in A \wr C^n$, we can find a quotient $\pi : C \to B$ such that $\hat{\pi} : A \wr C^n \to A \wr H$ maps g to a nontrivial element, and H is torsion-free nilpotent. Since A is torsion-free abelian and H is torsion-free nilpotent, $A \wr H$ is residually nilpotent by [10, Theorem B2], thus we can find a further quotient $\pi' : A \wr H \to N$ mapping g to a nontrivial element. Since g was arbitrary, $\pi' \circ \pi$ shows that $A \wr C^n$ is residually nilpotent. □

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