Gate lattices

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Abstract

A reversible gate on a subshift on a residually finite group G can be applied on any sparse enough finite-index subgroup H, to obtain what we call a gate lattice. Gate lattices are automorphisms of the shift action of H, thus generate a subgroup of the Hartman-Kra-Schmieding stabilized automorphism group. We show that for subshifts of finite type with a gluing property we call the eventual filling property, the subgroup generated by even gate lattices is simple. Under some conditions, even gate lattices generate all gate lattices, and in the case of a one-dimensional mixing SFT, they generate the inert part of the stabilized automorphism group, thus we obtain that this group is simple. In the case of a full shift this has been previously shown by Hartman, Kra and Schmieding.

1 Introduction

In this paper, we introduce a class of infinite simple groups based on the idea of applying "(reversible) gates", namely maps that modify only a bounded set of coordinates, simultaneously at various positions of an ambient group G. More precisely we apply them on translates of finite-index subgroups, or lattices, sparse enough so the applications commute. We call the resulting maps gate lattices, and we call the group they generate $\mathfrak{L}$.

As we show, in the case of the ambient group being $\mathbb{Z}$, this idea is in fact strongly connected to existing ideas in one-dimensional symbolic dynamics, in particular to the simple automorphisms of Nasu [10], and the inert automorphisms (kernel of Krieger's dimension representation [4, 3]). Namely, in the stabilized point of view explored in [7], these all turn out to be equivalent concepts on mixing SFTs.

The connection between simplicity and inertness is made in [14] (strengthened in [2]). In our terminology, we can essentially interpret this as saying that an inert automorphism can be split into finitely many applications of gate lattices. A related result is proved for two-dimensional full shifts in [8], namely it is shown that every automorphism is a product of finitely many block permutations and partial shifts. Block permutations are almost equivalent to our gate lattices, and partial shifts can be thought of as the device for making the automorphism inert (although we are not aware of a definition of inertness in this context). Some other papers exploring the idea of commuting (not necessarily reversible) gates are [13, 1, 5].

The observation that one-dimensional full shifts lead to a simple group of gate lattices is made in [7] (though not in this terminology). There the technical gist is the same as here: any nontrivial normal subgroup of $\mathfrak{L}$ actually contains a nontrivial gate lattice. This is Lemma 5.2 in [7].

Our proof of the analogous claim can be summarized in one sentence that does not really hide any technicalities: commutatoring any nice enough homeomorphism (in particular any element of the stabilized automorphism group) by a gate we obtain a gate, and performing the same commutatoring on a sufficiently sparse finite-index subgroup gives the corresponding gate lattice. Translating this into a proof takes some work, but most of this work is about setting up the algebra and basic theory of gate lattices.

1.1 Main statements

We now give the brief definition of our groups (see Section 2 for more detailed definitions) and a version of the main theorem statement.

Throughout this paper, G stands for a countable residually finite group. A set $X \subset \Sigma^G$ is a *subshift* if it is closed and shift-invariant (Σ a finite discrete alphabet), a *gate* on X is a homeomorphism that only modifies a bounded set of coordinates. A gate is *even* if the permutations on the finite set N it modifies are even in all possible contexts $x \in \Sigma^{G \setminus N}$. Our main object of interest is a group called $\hat{\mathfrak{L}}$ whose generators are homeomorphisms where even gates are applied on the right cosets of a finite index subgroup; we call these generators *even gate lattices*.

A *subshift of finite type* is a subshift (shift-invariant closed set) of the form $\bigcap_{g \in G} gC$ for $C \subset A^G$ clopen, which means the subshift is defined by a finite set of forbidden patterns. We also need to define a gluing property: a subshift has the *eventual filling property* if for any finite set F there exists a larger finite set N such that the patterns occurring in F are independent from those occurring in $G \setminus N$ (no control on the transition pattern in $N \setminus F$ is assumed).

Theorem 1. *Let X be a subshift of finite type with the eventual filling property, on an countably infinite residually finite group. Then the group generated by even gate lattices on X is simple.*

This is proved in Theorem 2.

In many cases, the group generated by even gate lattices is actually the same as the one generated by arbitrary gate lattices. In particular this happens whenever

- X has the property that in every *context* (pattern on a fixed co-finite set) the number of ways to fill the remaining hole is even;
- X is a full shift A^G, and G has *halvable subgroups*, meaning every finite index subgroup of G has a subgroup of even index;
- $G = \mathbb{Z}$.

In particular, full shifts have a simple $\mathfrak{L}$ if their alphabet is even, or the group has halvable subgroups. All finitely-generated infinite finite-dimensional matrix groups over commutative rings have halvable subgroups by [15], and so do (trivially) all infinite residually finite 2-groups, including the (non-linear) Grigorchuk group [6].

In the case $G = \mathbb{Z}$, the eventual filling property is equivalent to mixing, and in this case X is up to isomorphism the set of bi-infinite paths in a finite digraph whose associated matrix is primitive. In this case, $\hat{\mathfrak{L}}$ is not only equal to $\mathfrak{L}$, but is equal to the stabilized inert automorphism group of the subshift, as defined in [7]. This is shown in Proposition 1.

Corollary 1. *The stabilized inert automorphism group of a mixing one-dimensional subshift of finite type is simple.*

This result was proved in [7] for one-dimensional full shifts.

We also show that, in general, $\mathfrak{L}$ is normal as a subgroup of the stabilized automorphism group, so in the case $\hat{\mathfrak{L}} = \mathfrak{L}$, it is a *maximal* simple subgroup of the stabilized automorphism group. This is Theorem 3.

In particular, if one defines a notion of inertness for stabilized automorphisms groups on a group other than $\mathbb{Z}$, then if the group of stabilized inert automorphisms Q contains $\hat{\mathfrak{L}}$ (which we feel it should, as $\hat{\mathfrak{L}}$ is as "inert" as a group could possibly get), and we also happen to have $\mathfrak{L} = \hat{\mathfrak{L}}$, then Q is simple if and only if it is precisely $\hat{\mathfrak{L}}$. Thus, even though there is no standard notion of inertness for general groups, Theorem 1 can reasonably be considered a natural generalization of the result of [7] even beyond the one-dimensional case.

Our precise statement of Theorem 1 is Theorem 2, and it contains an extra generalization, namely we can take any net of finite-index subgroups (with enough normal subgroups), and consider only assignments of gate lattices where the lattice is on this net. One could define an analogous variant of the stabilized automorphism group by restricting to a subnet of finite-index subgroups, but our proof of Corollary 1 does not directly go through for arbitrary nets.

2 Setting the scene

2.1 Basic (but partially new) definitions

In this section we fix convensions, recall some basic definitions, and introduce some new ones. By $A \Subset B$ we mean A is a finite subset of B. For groups, our commutator convention is $[a, b] = a^{-1}b^{-1}ab$, and conjugation is $a^b = b^{-1}ab$. The identity element of a group G is 1_G. Intervals $[i, j]$ with $i, j \in \mathbb{Z}$ are discrete, i.e. $[i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$. By Σ^* we denote finite (possibly empty) words over alphabet Σ, i.e. elements of the free monoid, and for $u \in \Sigma^*$, $|u|$ denotes the length of u. In groups of homeomorphisms we write composition of ϕ_1 and ϕ_2 as $\phi_1 \circ \phi_2$ or simply $\phi_1\phi_2$, and the rightmost homeomorphism is applied first. The *support* of a homeomorphism is the smallest closed set such that every point outside is fixed.

A group G is *residually finite* if for every $g \neq e_G$, there exists a normal finite-index subgroup H such that $g \notin H$. We say G is a 2-group if every element has finite order which is a power of 2. By F_n we denote the free group on n generators. By S_n and A_n we denote the symmetric and alternating group on n elements, respectively, and $\mathrm{Sym}(A), \mathrm{Alt}(A)$ are the corresponding groups for a set A.

Throughout this paper (unless otherwise mentioned), G denotes a countably infinite residually finite discrete group, not necessarily finitely-generated. We think of this as an "ambient" group, and often omit it in statements (unless we need to specify further properties of it).

We order finite subsets of a set G by inclusion, and something holds for *arbitrarily large* subsets of G if for any finite set $F \Subset G$ it holds for some $S \supset F$. If G is finitely-generated, we say a sequence of translated finite-index subgroups $H_i g_i$ gets *arbitrarily sparse* if the distance between elements $hg_i, h'g_i$ is uniformly arbitrarily sparse in $h \in H_i$ in some right-invariant word metric; equivalently, the word norm of the minimal non-identity element in H_i grows without bound. On subsets of a group G we use the *Fell topology*, namely the Cantor topology $\{0,1\}^G$ after identifying sets with their characteristic functions. For the subspace of subgroups this is also known as the *Chabauty topology.* On a general group, we say $H_i g_i$ becomes arbitrarily sparse if $(H_i)_i$ tends to the trivial group.

We will frequently and without explicit mention use the following basic group theory fact, which works for any group G; for K one can simply take the kernel of the translation action on left cosets of H.

Lemma 1. *If $H \leq G$ is of finite index, then H contains a normal subgroup $K \triangleleft G$ of finite index.*

A *topological dynamical system* is a pair (X, G) where X is a compact metrizable space, G is a countable discrete group, and $G \curvearrowright X$ acts continuously on X. For $G = \mathbb{Z}$ we write this also just as (X, σ) where σ is the homeomorphism corresponding to the cyclic generator $1 \in \mathbb{Z}$. If G is a countable infinite group and Σ a finite set, we consider Σ^G with the product topology; topologically it is just the Cantor set. The group G acts on Σ^G by $(g, x) \mapsto \sigma_g(x)$ where $\sigma_g(x)_h = x_{hg}$. To shorten formulas, we often omit "σ" and identify elements of G with these translation maps. If X is a topologically closed G-invariant set $X \subset \Sigma^G$, (X, G) is called a *subshift* and $x \in X$ is called a *configuration.*

For $X \subset \Sigma^G$ and $D \subset G$, for $x \in X$ write $x|D \in \Sigma^D$ for the partial configuration $\forall g \in D : y_g = x_g$ (we do not drop the right argument of $|$ to a subscript, to avoid complex formulas in subscripts and double subscripting). Partial configurations are also called *patterns*, and they are *finite* if their domain is. Write $X|D \subset \Sigma^D$ for the set of patterns $x|D$ where $x \in X$. In the one-dimensional ($G = \mathbb{Z}$) situation for $u \in \Sigma^*$ we write $u \sqsubset X$ for $u \in X|[0, |u| - 1]$, with the obvious identification of words and patterns. When a subshift is clear from context, $y \in \Sigma^D$ is a pattern, and $N \subset G$, define $\mathcal{F}(y, N) = \{z \in \Sigma^N \mid \exists x \in X : x|D = y, x|N = z\}$. For two patterns $x \in \Sigma^D, y \in \Sigma^E$ with $D \cap E = \emptyset$, write $x \sqcup y$ for the obvious union pattern with domain $D \cup E$.

When a group G is clear from context, we fix a net $(B_r)_r$ of finite subsets of G that exhausts it (we do not name the directed set of rs). We call the finite set B_r the *ball* of radius r. For a symmetric set N, write $A_{r,N} = NB_r \setminus B_r$ for the *annulus* of *thickness* N. As the notation may suggest, we like to pretend our groups are finitely-generated; in this case one may fix some finite set of generators (all results will be independent of this choice, more generally the net). For d the corresponding right-invariant word metric, and $r \in \mathbb{N}$, we can pick $B_r = \{g \mid d(g, 1_G) \leq r\}$. Picking $N = B_R$, the corresponding annulus is just $A_{r,N} = B_{r+R} \setminus B_r$, explaining the terminology. Patterns whose domain is an annulus $A_{r,N}$ are often called *contexts*, and many of our arguments deal with the various fillings $x \in \mathcal{F}(P, B_r)$ (or $x \in \mathcal{F}(P, NB_r)$ if we want to keep the context visible) for contexts $P \in X|A_{r,N}$. Sometimes we use similar terminology with "full" contexts $x \in X|G \setminus N$ with $N \Subset G$.

The usual notion of isomorphism for topological dynamical systems is *topological conjugacy*, meaning homeomorphism commuting with the action. Up to topological conjugacy, we can define subshifts on countable groups as expansive actions on compact subsets of the Cantor set, where *expansivity* means that there exists $\epsilon > 0$ such that

$$(\forall g \in G : d(gx, gy) < \epsilon) \implies x = y$$

where d metrizes the Cantor topology.

A subshift of the form $\bigcap_{g \in G} gC$, where $C \subset \Sigma^G$ is clopen, is called an *SFT* (short for subshift of finite type). If $P \in \Sigma^D$ where $D \Subset G$, the *cylinder* defined by P is $[P] = \{x \in X \mid x|D = P\}$. A clopen set is a finite union of cylinders, and a *window* for an SFT X is any symmetric finite set N (meaning $N = \{g^{-1} \mid g \in N\}$) that contains all the sets DD^{-1} such that D appears among the cylinders. The important property of a window N is the following; we omit the straightforward proof.

Lemma 2. *If N is a window for X, then for all $y \in X|A_{r,N}$, the choices of followers $\mathcal{F}(y, B_r)$ and $\mathcal{F}(y, G \setminus NB_r)$ are independent in X, meaning for any $x \in \mathcal{F}(y, B_r)$ and $z \in \mathcal{F}(y, G \setminus NB_r)$, we have $x \sqcup y \sqcup z$ is in X.*

Definition 1. *A subshift X has the* eventual filling property*, or* EFP*, if*

$$\forall F \Subset G : \exists N \Subset G : \forall x, y \in X : \exists z \in X : z|F = x|F \wedge z|(G \setminus N) = y|(G \setminus N).$$

EFP is a *gluing property*, meaning it deals with compatibility of patterns on different areas of the group. On the groups $\mathbb{Z}^d$, the EFP property can be seen as a weakening of the uniform filling property, introduced in [12] for $\mathbb{Z}^2$s. To the best of our knowledge it has not been studied previously even on $\mathbb{Z}^2$.

As a technical weaker notion, we say a subshift has the *many fillings property*, or *MFP*, if as $F \Subset G$ tends to G, the number of configurations agreeing with $x|_{G \setminus F}$ tends to infinity uniformly in $x \in X$. A subshift is *nontrivial* if it has at least two configurations. The following is a simple exercise.

Lemma 3. *Every nontrivial EFP subshift has MFP, in particular it is infinite.*

It is also easy to see that a nontrivial EFP subshift cannot have isolated points (thus is homeomorphism to the Cantor set).

An *automorphism* of a subshift $X \subset \Sigma^G$ is a homeomorphism $f : X \to X$ that commutes with the action of G. By the definition of the topology on Σ^G, an automorphism has a (not necessarily unique) finite *neighborhood* $N \subset G$ and a *local rule* $\hat{f} : \Sigma^N \to \Sigma$ such that $f(x)_g = \hat{f}(P)$ where $P \in \Sigma^N$ is defined by $P_h = x_{hg}$. This is called the *Curtis-Hedlund-Lyndon theorem.* Obviously the inverse of an automorphism is an automorphism as well, so the automorphisms of a subshift form a group denoted by $\mathrm{Aut}(X, G)$. More generally, a topological conjugacy between two subshifts satisfies an obvious analog of the Curtis-Hedlund-Lyndon theorem.

Lemma 4. *For one-dimensional ($G = \mathbb{Z}$) SFTs, EFP is equivalent to the condition*

$$\exists m : \forall u, v \sqsubset X : \exists w : |w| = m \wedge uwv \sqsubset X.$$

(This is the one-dimensional case of the uniform filling property of [12].)

Proof. If the condition holds, then EFP is clear (even without the SFT assumption), namely for any finite set $F \subset [i, j]$ we can pick $N = [i - m, j + m]$ and use the condition on both sides of the interval to glue F-patterns (extended arbitrarily to $[i, j]$) to $(\mathbb{Z} \setminus N)$-patterns.

From Curtis-Hedlund-Lyndon, we easily see that topological conjugacy preserves both EFP and the condition in the lemma (even every *factor map*, i.e. surjective G-commuting continuous map, preserves them), so we can conjugate X to an *edge shift* [9], meaning X is the set of paths (sequences of edges with matching endpoints) in a finite directed graph (V, E). Let M be the matrix with $M_{a,b}$ the number of edges from vertex a to vertex b

Suppose now that we have EFP. Pick $F = \{0\}$ and N as in the definition of EFP. We can clearly make $G \setminus N$ smaller without breaking the gluing property for this pair, so we may take $N = [i, j]$ with $i \leq 0 \leq j$. Now in particular, by applying the gluing property to all pairs of particular configurations and looking at the rightmost vertex of the edge at 0, and the leftmost vertex of the edge at $j + 1$, we see that for any two vertices $a, b \in V$, we have a path of length j from a to b. Thus M^j is a matrix with all entries positive. Now the condition of the lemma holds with $m = j$. $\square$

The condition in the lemma is known to be equivalent to topological mixing [9], so we refer to it as simply *mixing*. In the proof we showed that EFP implies that X is defined, as an edge shift, by a *primitive* matrix, namely one with a positive power.

If $X \subset \Sigma^G$ is a subshift and $H \leq G$ is a finite index subgroup, then the action of the subgroup H on X can itself be considered a subshift, namely pick a set of left representatives R and define $\phi : \Sigma^G \to (\Sigma^R)^H$ by $(\phi(x)_h)_r = x_{rg}$. Alternatively, in terms of the abstract characterization of subshifts, it is easy to see that passing to a subgroup of finite index only changes the expansivity constant, since the G-action is continuous. Even if $X \subset \Sigma^G$ is not necessarily G-invariant, we say it is a K-subshift if it is closed and K-invariant.

A crucial property of automorphisms of subshifts is that they satisfy a dual notion of continuity, namely information cannot move from near the origin of the group to its ends in one step (which can happen with general homeomorphisms $f : \Sigma^G \to \Sigma^G$).

Definition 2. *Let G be a countable set, Σ an alphabet and $X \subset \Sigma^G$. A homeomorphism $f : X \to X$ is* ntinuous *if for each finite set S there exists a finite set F such that for $g \notin F$ and $x \in X$, $x \mapsto f(x)_g$ factors through the projection $x \mapsto x|G \setminus S$. A homeomorphism is* bintinuous *if it is ntinuous, and its inverse is also ntinuous.*

Not every homeomorphism is bintinuous, one example is the inverse of the map $f : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ defined by $f(x)_0 = x_0$ and $\forall i \geq 1 : f(x)_i \equiv x_{i-1} + x_i \bmod 2$. It is clear that automorphisms of subshifts (and thus also their inverses) are bintinuous. It is also clear that bintinuous homeomorphisms on a subshift X form a group.

We need a simple fact about uniform convergence. Note that we only deal with homeomorphisms on Cantor space, so all metrics are equivalent and all continuous functions are uniformly continuous.

Lemma 5. *Let X, Y, Z be metric spaces and let $g_i, g : Y \to Z, f_i, f : X \to Y$ be functions, where i runs over some directed set $\mathcal{I}$. If $g_i \to g$ and $f_i \to f$ uniformly and g is uniformly continuous, then $g_i \circ f_i \to g \circ f$ uniformly.*

Proof. Let $\epsilon > 0$ be arbitrary. Let j_0 be such that for $j \geq j_0$ we have $\forall y \in Y : d_Z(g_j(y), g(y)) < \epsilon/2$. Use uniform continuity of g to find $\delta > 0$ such that $\forall x, y \in Y : d_Y(x, y) < \delta \implies d_Z(g(x), g(y)) < \epsilon/2$. Use uniform convergence of $(f_i)_i$ to find $i_0 \geq j_0$ such that for $i \geq i_0$ we have $\forall x \in X : d_Y(f_i(x), f(x)) < \delta$.

Suppose now that $i \geq i_0$ and let $x \in X$ be arbitrary. We have

$$d_Z(g(f(x)), g(f_i(x))) < \epsilon/2$$

because $d_Y(f(x), f_i(x)) < \delta$ and by the choice of δ, and we have

$$d_Z(g(f_i(x)), g_i(f_i(x))) < \epsilon/2$$

because $i \geq j_0$, so by the triangle inequality we have

$$d_Z(g(f(x)), g_i(f_i(x))) \leq d_Z(g(f(x)), g(f_i(x))) + d_Z(g(f_i(x)), g_i(f_i(x))) < \epsilon$$

proving uniform convergence of $g_i \circ f_i$ to $g \circ f$. $\square$

2.2 Gates

Definition 3. *A* gate *on a subshift $X \subset A^G$ is a homeomorphism $\chi : X \to X$ such that for some* weak neighborhood *$N \Subset G$ we have $\chi(x)_g = x_g$ for all $g \notin N$. Write $\mathfrak{G}$ for the group generated by gates.*

Note that "$\mathfrak{G}$" is a fancy "G", and stands for "gate". The following lemma was proved for $G = \mathbb{Z}$ in [13]; the proof for general G is the same, and follows more or less directly from the definition of the product topology.

Lemma 6. *A homeomorphism $\chi : X \to X$ is a gate if and only if it admits a* strong neighborhood *$N \Subset G$ such that for some* local rule *$\hat{\chi} : X|N \to X|N$ such that for all $g \notin N$ we have $\chi(x)_g = x_g$ for all $x \in X$, and for all $g \in N$ we have $\chi(x)_g = \hat{\chi}(x|_N)_g$.*

It is clear that $\mathfrak{G}$ in fact consists of gates, as we can always increase the strong neighborhoods of two gates to be equal (after which they compose like permutations). For a gate χ and $g \in G$ write $\chi^g = \sigma_{g^{-1}} \circ \chi \circ \sigma_g$. Note in particular that when $G = \mathbb{Z}$, χ^n does not refer to iteration – we never need to iterate a gate. One can see χ^g as applying the gate "at" g, if we use the convention where configurations of X are seen as vertex-labelings of some left Cayley graph of the group G (at least when it is finitely-generated). If χ has strong (resp. weak) neighborhood N, then χ^g has strong (resp. weak) neighborhood Ng.

Gates form a normal subgroup of the group of bintinuous homeomorphisms on X:

Lemma 7. *Let f be a homeomorphism with ntinuous inverse and χ a gate. Then χ^f is a gate.*

Proof. Suppose that $N \Subset G$ is a strong neighborhood for χ. By ntinuity of f^{-1}, outside some finite set F, the image of f^{-1} can be determined without looking at the cells in N, in other words for $g \notin F$, we have $f^{-1}(\chi(f(x)))_g = f^{-1}(f(x))_g = x_g$ since the application of χ does not affect the f^{-1}-image. This shows that F is a weak neighborhood for χ^f. $\square$

We say two gates χ, χ' *commute* if they do, i.e. if $[\chi, \chi'] = \chi^{-1} \circ (\chi')^{-1} \circ \chi \circ \chi' = \mathrm{id}$. It is clear that if χ, χ' have strong radii N, N' respectively, and $N \cap N' = \emptyset$, then χ and χ' commute. If $S \subset G$ is a (possibly infinite) subset, and χ^s commutes with χ^t for any distinct $s, t \in S$, then we say *the product* $\prod_{s \in S} \chi^s$ *commutes*, or just that χ^S *commutes*, and define

$$\chi^S(x) = \lim_{F \Subset S} (\prod_{k \in F} \chi^k)(x)$$

by taking the pointwise limit (note that $\prod$ means function composition here).

Lemma 8. *If χ^K commutes, then the pointwise limit is well-defined, and convergence is uniform.*

Proof. To see that the pointwise limit is well-defined, observe that in a finite subproduct χ^F, $F \Subset K$, the value of $\chi^F(x)_g$ only depends on the value of finitely many elements of G, since we may order the product so that translates of χ that may change the value of g (i.e. g is in their strong neighborhood) are applied first. The same argument gives uniform convergence: the value at g stabilizes once we have applied all translates of χ with g in their strong neighborhood. □

Say a gate is *eventually even* if for all large enough strong neighborhoods N the corresponding permutation $\pi \in \mathrm{Sym}(X|N)$ restricted to any complement pattern is even, i.e. for any $y \in X|G \setminus N$, the restriction of π to the set $\mathcal{F}(y, N)$ is even. Say a gate is *sometimes even* if the same is true for at least one strong neighborhood N.

Lemma 9. *On every subshift, each of the following implies the next.*

1. $\chi = [\chi_1, \chi_2]$ *for some gates* χ_1, χ_2*;*
2. $\chi \in [\mathfrak{G}, \mathfrak{G}]$*;*
3. χ *is eventually even;*
4. χ *is sometimes even.*

The last two items are always equivalent, and on an MFP SFT all four are equivalent.

Due to the last point, we simply call eventually/sometimes even gates *even*. Write $\hat{\mathfrak{G}}$ for the group generated by even gates (we only consider MFP SFTs, so one can read $\hat{\mathfrak{G}}$ as $[\mathfrak{G}, \mathfrak{G}]$, but the concrete characterization is more useful).

Proof. The implication (1) $\implies$ (2) is trivial. For (2) $\implies$ (3) take a any N larger than the radius of all the gates involved in a composition of commutators of gates. For any $y \in X|G \setminus N$, the permutation of $\mathcal{F}(y, N)$ performed in the context is just the corresponding composition of commutators of gates restricted to $\mathcal{F}(y, N)$, and thus is in the commutator subgroup of the symmetric group of that set, which is the corresponding alternating group. The implication (3) $\implies$ (4) is trivial.

We show (4) $\implies$ (3) in every subshift, so (3) and (4) are equivalent. Simply observe that if N is a strong neighborhood such that χ performs an even permutation on N in every $(G \setminus N)$-context, then the same is true for any $(G \setminus N')$-context for $N' \supset N$, as for any $(G \setminus N')$-context we can write the permutation χ performs on $\mathcal{F}(y, N')$ as a finite composition of even permutions. Namely, for each of the finitely many extensions $z \in \mathcal{F}(y, G \setminus N)$ the permutation χ performs on the pattern in N is even by assumption.

It now suffices to show that in an MFP SFT X, (3) $\implies$ (1). To see this, let N be a window for X and pick a large strong neighborhood B_r such that there are at least 5 fillings of each $A_{r,N}$-context. Now increasing the strong neighborhood to NB_r, we have a permutation of $X|NB_r$ which does not modify the contents of the annulus $A_{r,N}$ and for each pattern on the annulus performs an even permutation on the pattern inside. Since there are at least 5 extensions of the pattern and $[S_n, S_n] = \{[g, h] \mid g, h \in S_n\}$ for $n \geq 5$ [11], we can write the restriction to each $A_{r,N}$-context as a commutator of two permutations in that same context. For different contexts the permutations commute, so we can write this as a commutator of two gates. □

2.3 Gate lattices

Lemma 10. *Suppose that $K \leq G$ is a subgroup, $X \subset \Sigma^G$ is a K-subshift, χ is a gate on X, and χ^K commutes. Then χ^K is an automorphism of (X, K).*

Proof. To see that χ^K is a homeomorphism, observe that by Lemma 8 it is a uniform limit of continuous functions, thus continuous (alternatively, the proof shows this directly). It clearly has continuous inverse $(\chi^{-1})^K$, thus it is a homeomorphism.

We check commutation with K-shifts. If $h \in K$, then

$$\begin{aligned}
\chi^K \circ \sigma_h &= \lim_{F \Subset K} (\prod_{k \in F} \chi^k) \circ \sigma_h \\
&= \lim_{F \Subset K} \prod_{k \in F} \sigma_{k^{-1}} \circ \chi \circ \sigma_k \circ \sigma_h \\
&= \lim_{F \Subset K} \prod_{k \in F} \sigma_h \circ \sigma_{(kh)^{-1}} \circ \chi \circ \sigma_{kh} \\
&= \lim_{F \Subset K} \prod_{k \in F} \sigma_h \circ \sigma_{k^{-1}} \circ \chi \circ \sigma_k \\
&= \sigma_h \circ \chi^K
\end{aligned}$$

where uniform convergence of products means limits commute with composition (Lemma 5), and the fourth equality holds because Fh runs over the finite subsets of K as F does. □

Lemma 11. *Suppose X is a G-subshift, $g \in G$ and $H \leq G$. If χ^H commutes, then χ^{Hg} commutes and $\chi^{Hg} = (\chi^H)^g$.*

Proof. Interpreting infinite products as pointwise uniform limits of finite products and applying Lemma 5 to pull functions out of the limit, the calculation

$$\chi^{Hg} = \prod_{h \in H} \sigma_{g^{-1}h^{-1}} \circ \chi \circ \sigma_{hg} = (\prod_{h \in H} \sigma_h \circ \chi \circ \sigma_{h^{-1}})^g = (\chi^H)^g$$

suffices. □

Example 1: We note that the product χ^{gH} may not commute even if χ^H commutes. Suppose e.g. that $G = F_2 = \langle a, b \rangle$, $X = \{0,1\}^G$, $H = \langle a \rangle$ and χ swaps the symbols at $\{b, ba\}$. Clearly χ^H commutes, but $\chi^{b^{-1}H}$ does not. ○

Definition 4. *Maps of the form χ^{Hg} where $H \leq G$ is of finite index are called* gate lattices*, and they are* even *if χ is. Write $\mathfrak{L}$ for the group generated by gate lattices, and $\hat{\mathfrak{L}}$ for the group generated by even gate lattices.*

Note that "$\mathfrak{L}$" is a fancy "L", and stands for "lattice", which refers to the fact gates are applied at the points of a (translated) lattice.

We do not need to take *all* subgroups of finite index in this definition for some of our results, in particular in the precise statement of the main result Theorem 2 we allow any net of finite-index subgroups with a cofinal subnet of normal subgroups, tending to the trivial group in the Chabauty topology (presumably leading to different groups for different such nets). Such a net $\mathcal{I}$ will be called a *lattice net*; for simplicity we directly take $\mathcal{I}$ to be a set of finite-index subgroups ordered under inclusion and directed downward. We define $\mathcal{I}$-gate lattices in the obvious way, as well as notations $\mathfrak{L}_{\mathcal{I}}, \hat{\mathfrak{L}}_{\mathcal{I}}$. Note that of course $\mathfrak{L}_{\mathcal{I}}, \hat{\mathfrak{L}}_{\mathcal{I}}$ are respectively subgroups of $\mathfrak{L}, \hat{\mathfrak{L}}$.

Lemma 12. *Suppose χ^{Hg} commutes, and $K \leq H$. Then χ^{Hg} can be written as a finite commuting product of commuting gates of the form $\chi^{Kg'}$*

Proof. Simple write $H = \bigcup_{h \in T} Kh$ for some set of representatives T, and observe that

$$\prod_{h \in H} \chi^{hg} = \prod_{h \in T} \prod_{k \in K} \chi^{khg} = \prod_{h \in T} \chi^{Khg}$$

by the commutation of χ^{Hg}. □

Lemma 13. *If H is a normal subgroup of G, then χ^{Hg} is an automorphism of (X, H) whenever χ^H commutes.*

Proof. we have $\chi^{Hg} = \chi^{gH}$ by normality, and the latter product, equal to $(\chi^g)^H$, must commute since by Lemma 11 χ^{Hg} does, and gH and Hg are literally the same subset of G where χ gets applied. Now $\chi^{Hg} = (\chi^g)^H$ is an automorphism of (X, H) by Lemma 10. □

We recall a definition of Hartman-Kra-Schmieding (generalized to residually finite acting groups, as seems appropriate here).

Definition 5. *Let G be a residually finite group, and $X \subset \Sigma^G$ a subshift. The* stabilized automorphism group *of X is the direct union of $\mathrm{Aut}(X, H)$ where H ranges over finite-index subgroups of G.*

Lemma 14. *The groups $\mathfrak{L}$ and $\hat{\mathfrak{L}}$ are contained in the stabilized automorphism group of Σ^G.*

Proof. Every finite-index subgroup of contains a normal finite-index subgroup. By Lemma 12 we can write any χ^{Hg} as a composition of maps of the form $\chi^{Kg'}$ with normal K, and by Lemma 13 these are automorphisms of (X, K), thus in the stabilized automorphism group. □

2.4 Conditions on gate lattices being even

Lemma 15. *If X is an MFP SFT, then $\hat{\mathfrak{L}}$ is contained in $[\mathfrak{L}, \mathfrak{L}]$.*

Proof. Let $\chi^{Hg} \in \hat{\mathfrak{L}}$. By Lemma 9 we have $\chi = [\chi_1, \chi_2]$ for some gates χ_1, χ_2. Using Lemma 12 and residual finiteness of G, we may assume H is very sparse compared to the strong neighborhood of χ and the χ_i. Then an easy calculation shows $\chi^{Hg} = [\chi_1^{Hg}, \chi_2^{Hg}]$. Namely the strong neighborhoods of different translates of χ and χ_i do not intersect, thus these translates commute, thus for finite $F \Subset H$ we have

$$\begin{aligned}\chi^{Fg} &= \prod_{h \in F} (\chi_1^{-1})^{hg} \circ (\chi_2^{-1})^{hg} \circ \chi_1^{hg} \circ \chi_2^{hg} \\ &= \prod_{h \in F} (\chi_1^{-1})^{hg} \circ \prod_{h \in F} (\chi_2^{-1})^{hg} \circ \prod_{h \in F} \chi_1^{hg} \circ \prod_{h \in F} \chi_2^{hg} \\ &= [\chi_1^{Fg}, \chi_2^{Fg}]\end{aligned}$$

and the claim follows by taking the (uniformly converging) limits and applying Lemma 5. □

We do not know when $[\mathfrak{L}, \mathfrak{L}]$ is equal to $\hat{\mathfrak{L}}$, but we show some conditions under which we even have $\hat{\mathfrak{L}} = \mathfrak{L}$.

We first show that a full shift over an even alphabet over any group G has this property. We in fact state some more general dynamical properties that imply it. First say a subshift has *even fillings* if for some $F \Subset G$, we have that $|\mathcal{F}(x, F)|$ is even for all $x \in X|G \setminus F$. It is clear that if this happens for some F, it happens for all larger F and all right translates of F.

Lemma 16. *If X is an MFP SFT with even fillings, then $\hat{\mathfrak{L}}_{\mathcal{I}} = [\mathfrak{L}_{\mathcal{I}}, \mathfrak{L}_{\mathcal{I}}] = \mathfrak{L}_{\mathcal{I}}$ for any lattice net $\mathcal{I}$.*

Proof. If χ has strong neighborhood N, let R be a window for X, let F have even fillings, and replace the strong neighborhood of χ by $N \sqcup Fg$ such that $N \cap RFg = \emptyset$, acting trivially on the contents of Fg. Now consider any context $x \in X|G \setminus (N \cup Fg)$. Since applying χ does not affect the contents of RFg, the possible contents of Fg only depends on x before and after the application.[1] Thus for each even cycle that χ performs, we perform an even number of independent copies of it when χ is seen as a permutation of $\mathcal{F}(x, N \sqcup Fg)$, in particular this is an even permutation. It is clear that χ with the new neighborhood still commutes with its translates (since it is the same homeomorphism). □

A common way to get even fillings is that we literally have freely changeable bits visible on each configuration. More precisely, we say a subshift has *syndetic free bits* if it is conjugate to a subshift X over the disjoint union alphabet $A \cup (B \times \{0, 1\})$ such that elements of $B \times \{0, 1\}$ appear in every configuration, and if $x \in X$ and $x_g = (b, c) \in B \times \{0, 1\}$, then also $y \in X$, where $y_h = x_h$ for $h \neq g$ and $y_g = (b, 1 - c)$.

Lemma 17. *If X has syndetic free bits, then is has even fillings.*

[1] Since χ is well-defined with strong neighborhood N, it is not even necessary to assume $N \cap RFg = \emptyset$, only that $N \cap Fg = \emptyset$; but the extra leeway does not hurt.

Proof. A free involution on the fillings of a hole F is obtained by flipping the first free bit under any ordering of F (and a free involution is a bijective proof of even cardinality). The hole F simply needs to be large enough so there is always a free bit. Indeed, since free bits appear in all configurations of X, by compactness they appear in any large enough finite pattern of any configuration, thus such F exists. □

Next, we cover all full shifts, under a condition on the group. We say a group G *has halvable subgroups* if all of its finite index subgroups have a finite index subgroup with an even index, i.e. $\forall H \leq G : \exists K \leq H : 2|[H : K]$.

Lemma 18. *If G has halvable subgroups and $X = \Sigma^G$, then $\hat{\mathfrak{L}} = [\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.*

Proof. It suffices to write any χ^{Hg} as a commutator, where H is a sufficiently sparse normal finite index subgroup. Let $K \leq H$ have even index, take right coset representatives $k_1, ..., k_{2n} \in H$. By normality of K we have $\chi^{Kk_i} = \chi^{k_i K}$ and we can see χ^{Hg} as a composition of maps $(\chi^{k_{2i+1}} \circ \chi^{k_{2i+2}})^{Kg}$.

Clearly $\chi^{k_{2i+1}}$ and $\chi^{k_{2i+2}}$ have the same parity in any annular context because the subshift's coordinates are interchangeable (we are on a full shift). Thus the parity of their composition is even in every context. By MFP and Lemma 9 we can write their product as a commutator. □

We give two examples of (classes of) groups with halvable subgroups.

Lemma 19. *The following groups have halvable subgroups:*

- *all finitely-generated infinite finite-dimensional matrix groups over commutative rings;*
- *all infinite residually finite 2-groups.*

Proof. The first item is a special case of a more general result of [15]. For the latter, it is clearly enough to prove that every infinite residually finite 2-group has a subgroup of even finite index. Take any proper normal subgroup $N \triangleleft G$ and consider the group G/N. The order of any nontrivial element gN is a power of 2, so by Lagrange's theorem the order $[G : N]$ of G/N is even. □

Of course one could generalize Lemma 18 to the case $\mathfrak{L}_{\mathcal{I}}$, namely simply take it as a property of a lattice net $\mathcal{I}$ that any $H \in \mathcal{I}$ has an even index subgroup $K \in \mathcal{I}$. Note that even $\mathbb{Z}$ does not have this property for all lattice nets $\mathcal{I}$.

In Lemma 18, we could also replace the condition of being a full shift with a weaker statement, namely that that number of connecting patterns between an annular pattern and another pattern positioned in two different fixed areas inside it have the same parity (with suitable quantifiers). We omit a precise statement, but in the case of $\mathbb{Z}$ we can use this idea to generalize the statement $\hat{\mathfrak{L}} = \mathfrak{L}$ to all mixing SFTs.

Lemma 20. *If $G = \mathbb{Z}$ and X is a mixing SFT, then $\hat{\mathfrak{L}} = [\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.*

Proof. It clearly suffices to show that, if χ is an arbitrary permutation with strong neighborhood $[0, n-1]$, then $\chi^{n\mathbb{Z}}$ is a composition of even gate lattices. We may assume X is represented as an edge shift, so elements of $\mathbb{Z}$ carry edges of a finite directed graph and there are vertices between them at elements of $\mathbb{Z} + \frac{1}{2}$; let M be the matrix with $M_{a,b}$ the number of edges from vertex a to vertex b. Note that the permutation π that χ performs in $[0, n-1]$ fixes the vertices at $-\frac{1}{2}$ and $n - \frac{1}{2}$. Write $p_{a,b}$ for the parity of the restriction of π to the context where the vertices at $(-\frac{1}{2}, n - \frac{1}{2})$ are respectively (a, b).

Next let $\hat{M}$ be the matrix obtained by taking entries of M modulo 2, and observe that $\hat{M}^n = \widehat{M^n}$. Observe that powers of $\hat{M}^n$ eventually get into a cycle, meaning $\hat{M}^{mn} = \hat{M}^{(m+p)n}$ for some $m \geq 0, p \geq 1$. Now consider the commuting product $\chi \circ \chi^{\sigma_{pn}}$. We claim that if we use the strong neighborhood $[-mn, mn + pn - 1]$, this is an even permutation.

To see this, let $a, b \in V$ and consider the permutation χ with vertices (a, b) at $(-mn-\frac{1}{2}, mn+pn-\frac{1}{2})$. For $c, d \in V$, for every choice of path from a to c and from d to b, counting modulo 2, χ has $p_{c,d}$ cycles of even length. If the number of paths from a to c is even or the number of paths from d to b is even, these cancel out, so the parity is just the parity of then number of triples (a, c, d, b) such that $\hat{M}^{mn}_{a,d} = p_{c,d} = M^{(m+p)n}_{d,b} = 1$. The same calculation holds for $\chi^{\sigma_{pn}}$, so their composition performs an even permutation in the context (a, b).

We conclude that $\chi \circ \chi^{\sigma_{pn}}$ with a suitable choice of strong neighborhood is even, and clearly the choice of strong neighborhood does not affect the commutation of the product $(\chi \circ \chi^{\sigma_{pn}})^{2np\mathbb{Z}}$. By applying χ in a similar paired-up way on other cosets of $pn\mathbb{Z}$, we get $\mathfrak{L} \subset \hat{\mathfrak{L}}$. Since $\hat{\mathfrak{L}} \subset [\mathfrak{L}, \mathfrak{L}] \subset \mathfrak{L}$, we conclude that the three groups are equal. □

It seems that the above proof essentially uses the fact $\mathcal{I}$ is the net of all finite-index subgroups.

2.5 Inertness

In the case $G = \mathbb{Z}$, there is a standard notion of *inertness* for an automorphism of a mixing SFT, meaning an automorphism that acts trivially on Krieger's *dimension group* (see e.g. [4, 3]). We omit the general definition of inertness, and only use a result of Wagoner [14] (except for the proof of the well-known fact (2) $\implies$ (1) below), namely Lemma 21 below. To state it, we need a few definitions.

Recall again that an edge shift is the set of paths $p : \mathbb{Z} \to E$ (with matching endpoints for successive edges) where (V, E) is a directed (multi-)graph (with loops). A *simple graph symmetry* is an automorphism of an edge shift which is defined by a bijection $\pi : E \to E$ that preserves the tails and heads of all vertices. If X is an SFT, an automorphism $f \in \mathrm{Aut}(X, \sigma)$ is *simple* if there exists a topological conjugacy between (X, σ) and an edge shift, so that f is a simple graph symmetry. This definition is due to Nasu [10]. The relevant result of Wagoner is the following:

Lemma 21. *If f is an inert automorphism of a mixing SFT (X, σ), then there exists $m \in \mathbb{Z}_+$ such that for all $n \geq m$, f can be written as a product of simple automorphisms of* $\mathrm{Aut}(X, \sigma^n)$.

Lemma 22. *Let X be a one-dimensional mixing SFT. Then the following are equivalent:*

1. *f is a product of inert automorphisms of (X, σ^n) for some $n \in \mathbb{Z}_+$;*
2. *f is a product of simple automorphisms of (X, σ^n) for some $n \in \mathbb{Z}_+$;*

3. *f is a product of gate lattices on X;*

4. *f is a product of even gate lattices on X.*

Proof. (1) $\implies$ (2): If f is inert for some (X, σ^n), then because (X, σ^n) is itself topologically conjugate to a mixing SFT, by Wagoner's result it is a product of simple automorphisms of higher powers of σ^n.

(2) $\implies$ (1): This is obvious from the definition of the dimension group through rays and beams [9], namely if we represent X as the edge where the automorphism is a simple graph symmetry, the action on rays is clearly the identity map.

(2) $\implies$ (3): If f is a simple automorphism of σ^n, then the edge permutations are obviously commuting gates, and thus f is directly a gate lattice on the subgroup $n\mathbb{Z}$ (note that looking through a topological conjugacy of course does not change the set of gates, nor affect the commutation of their translates).

(3) $\implies$ (2): We simply need to show that if $\chi^{n\mathbb{Z}}$ is a gate lattice, then it can be written as a product of simple automorphisms. This is straightforward from Lemma 12, namely we can write $n\mathbb{Z}$ as a union of sparser translated subgroups $jn + mn\mathbb{Z}$, and it is easy to see that for large enough m, $\chi^{jn+mn\mathbb{Z}}$ is a simple automorphism of $(X, \sigma^{mn\mathbb{Z}})$: One can take the vertices to represent words of length r around the positions $j + \lfloor mn/2 \rfloor + kmn$, where $[0, r-1]$ is a window for X. For large m, these words are not modified by $\chi^{j+mn\mathbb{Z}}$, and indeed if the strong neighborhood of χ is contained in $[-mn/2 + r, mn/2 - r]$ then χ can be seen as a permutation of the edges.

The equivalence of (3) and (4) is Lemma 20. □

If $G = \mathbb{Z}$ and X is a mixing SFT, the *stabilized inert automorphism group* is the smallest group containing the inert automorphisms of the mixing SFTs (X, σ^n), for all $n \geq 1$. The following is immediate from Lemma 22.

Proposition 1. *The stabilized inert automorphism group of a topologically mixing $\mathbb{Z}$-SFT is equal to its group $\hat{\mathfrak{L}}$.*

3 Proof of main results

We now prove the main results Theorem 2 and Theorem 3. The structure of the proof Theorem 2 is the following:

1. We commutator an arbitrary element of $\hat{\mathfrak{G}}$ (really any nontrivial homeomorphism) with a suitably chosen element of $\hat{\mathfrak{G}}$ to obtain a non-trivial map (first three lemmas).

2. We observe that applying the same on a lattice gives us a nontrivial gate lattice (Lemma 26).

3. We observe that once we have one gate on sparse enough lattices, by EFP we have all of them on sparse enough lattices (done in the proof of Lemma 28).

4. Once we can apply arbitrary gates on sparse enough lattices, we can apply them on any lattice by refinement (done in the proof of Lemma 28).

Lemma 23. *For $H \leq G$ of finite index, all elements of* $\mathrm{Aut}(X, H)$ *are bintinuous. In particular, elements of* $\mathfrak{L}$ *are bintinuous.*

Proof. Since (X, H) is a subshift, this is immediate from the Curtis-Hedlund-Lyndon theorem. □

Lemma 24. *Let f be a homeomorphism with ntinuous inverse and χ a gate. Then $[f, \chi]$ is a gate.*

Proof. To see that $[f, \chi] = f^{-1} \circ \chi^{-1} \circ f \circ \chi$ is a gate, it suffices to show that $(\chi^{-1})^f$ is a gate, and since χ is arbitrary it suffices to show that χ^f is. This is Lemma 7. □

Lemma 25. *Let f be a nontrivial homeomorphism with ntinuous inverse. If X is an SFT with MFP then $[f, \chi]$ is a nontrivial gate, for some gate $\chi \in \hat{\mathfrak{G}}$.*

Proof. Let R be a window for X. Let $x \in X$ be such that $f(x)_g \neq x_g$ for some $g \in G$, and pick a large enough r so that any pattern P on the annulus $A_{r,R}$ has at least 3 fillings for B_r. By gates in $\hat{\mathfrak{G}}$ we can realize any even permutation of $\mathcal{F}(P, RB_r)$ for any $P \in X|A_{r,R}$. Now pick χ any nontrivial 3-rotation of patterns in $\mathcal{F}(x|A_{r,R}, RB_r)$ which has x in its support, but not $f(x)$. If $x|A_{r,R} \neq f(x)|A_{r,R}$, this is trivial, otherwise it follows from $|\mathcal{F}(x|A_{r,R}, B_r)| \geq 3$, since of the at least three different fillings, at most one can agree with $f(x)$. Now f cannot commute with χ, as it does not preserve its support. □

Lemma 26. *Let $f \in \mathrm{Aut}(X, H)$ for some $H \leq G$ of finite index, and χ a gate. If $K \leq H$ is sparse enough, then we have $[f, \chi]^K = [f, \chi^K]$ (and these expressions are well-defined).*

Proof. Since f is H-invariant, it is bintinuous. Since $[f, \chi]$ is a gate, and f is also K-invariant, we have $f^k = f, (f^{-1})^k = f^{-1}$, so

$$[f, \chi]^K = \prod_{k \in K} [f, \chi]^k = \prod_{k \in K} f^{-1} \circ (\chi^{-1})^k \circ f \circ \chi^k$$

is well-defined for sparse enough K.

Since f is H-invariant and $K \leq H$, for sparse enough K, χ^k commutes with $((\chi^{-1})^{k'})^f$ whenever $k \neq k'$. Namely, we have

$$((\chi^{-1})^{k'})^f = f^{-1} k'^{-1} \chi^{-1} k' f = k'^{-1} f^{-1} \chi^{-1} f k' = ((\chi^{-1})^f)^{k'},$$

and it follows from Lemma 7 that $(\chi^{-1})^f$ admits a strong neighborhood N'. If N is the strong neighborhood of χ, it suffices that $Nk \cap N'k' = \emptyset$ for $k \neq k'$, which is just the condition $k'k^{-1} \notin N'^{-1}N$ and holds for sparse enough K.

Thus, finite subproducts of $[f, \chi]^K$ can be rearranged to approximations of $[f, \chi^K]$, in the sense that if $F \subset K$ is finite then

$$\begin{aligned} \prod_{k \in F} f^{-1} \circ (\chi^{-1})^k \circ f \circ \chi^k &= \prod_{k \in F} ((\chi^{-1})^k)^f \circ \chi^k \\ &= \prod_{k \in F} ((\chi^{-1})^k)^f \circ \prod_{k \in F} \chi^k \\ &= [f, \chi^F], \end{aligned}$$

thus the two infinite products are also the same. □

Lemma 27. *For any $n \geq 5$ and $f \in S_n \setminus \{1_{S_n}\}$, the smallest subgroup of S_n containing f and invariant under conjugation by A_n contains A_n.*

Proof. If $f \in S_n \setminus A_n$ is not the identity permutation, observe that f cannot commute with every 3-cycle, as it does not preserve the support of every 3-cycle, and thus we find a nontrivial commutator between f and f^π for a 3-cycle π. Thus we have generated a nontrivial element of A_n. If $f \in A_n$, the claim is clear since A_n is simple. □

Theorem 1 is obtained as an immediate consequence of the following lemma.

Lemma 28. *Let X be a EFP SFT on a residually finite group G, and let $f \in \mathrm{Aut}(X, H)$ be nontrivial where $H \leq G$ is of finite index. Then*

$$\langle f^{\hat{\mathfrak{L}}_{\mathcal{I}}} \rangle = \langle f, \hat{\mathfrak{L}}_{\mathcal{I}} \rangle$$

for any lattice net $\mathcal{I}$ containing H.

In words, the conclusion is that the smallest subgroup of $\mathrm{Homeo}(X)$ which contains all conjugates of f by gate lattices actually contains all gate lattices.

Proof. Take an arbitrary nontrivial element $f \in \mathrm{Aut}(X, H)$. By Lemma 23, f is bintinuous. Since f it is nontrivial, by Lemma 25, $[f, \chi]$ is a nontrivial gate for some $\chi \in \hat{\mathfrak{G}}$. Writing $\chi_0 = [f, \chi]$, we have by Lemma 24 that $\chi_0^K = [f, \chi^K]$ for any sparse enough $K \in \mathcal{I}$. Since $[f, \chi^K] = f^{-1} f^{\chi^K}$, we have that $\langle f^{\hat{\mathfrak{L}}_{\mathcal{I}}} \rangle$ contains these gate lattices χ_0^K.

We next show that from these, one can generate all even gate lattices χ^K where $K \leq H$, $K \in \mathcal{I}$. Let R be a window for X, let N be a strong neighborhood such that χ_0 performs a nontrivial permutation of $X|N$, and large enough so that the cardinality of the latter set is at least 5. If we pick a large enough r, then every $A_{r,R}$-context $P \in X|A_{r,R}$ allows an extension to any pattern in N by EFP. Thus with the strong neighborhood RB_r, χ_0 performs a permutation that fixes the $A_{r,R}$-context and acts nontrivially on the B_r-continuation for any context.

In a fixed such context $P \in X|A_{r,R}$ we can represent any even permutation of its extension patterns to RB_r (i.e. the set $\mathcal{F}(P, RB_r)$) as a composition of conjugates of χ_0 by commutators with even permutations that fix the context P, by Lemma 27. Composing these representations over all contexts, we can represent any even gate as a commutator expression involving only $\hat{\mathfrak{G}}$-conjugates of χ_0. Doing this simultaneously on all positions of a sparse enough finite-index subgroup $K \in \mathcal{I}$, we obtain for any even gate χ all elements of the form χ^K for all sparse enough $K \in \mathcal{I}$.

Consider now $K \leq H$, $K \in \mathcal{I}$ and suppose χ is an even gate and K is sparse enough so that χ^K can be built with the above construction. Observe that since $f \in \mathrm{Aut}(X, H)$ we have $f^h = f$ for all $h \in H$, and thus if we conjugate the expression for χ^K (which is a composition of conjugates of f by elements of $\hat{\mathfrak{L}}_{\mathcal{I}}$), the h-translation only affects the $\hat{\mathfrak{L}}_{\mathcal{I}}$-elements, so actually we obtain χ^{Kh} for cosets of K in H.

To say the same in formulas, if

$$\chi^K = \prod_i f^{\chi_i^{K_i}}$$

(note that this is a finite ordered product; actually $K_i = K$ for all i in our construction, but this is immaterial) then using Lemma 11 we have

$$\begin{aligned}\prod_i f^{\chi_i^{K_i h}} &= \prod_i (\chi_i^{-1})^{K_i h} \circ f \circ \chi_i^{K_i h} \\ &= \prod_i ((\chi_i^{-1})^{K_i})^h \circ f^h \circ (\chi_i^{K_i})^h \\ &= \prod_i (f^{\chi_i^{K_i}})^h \\ &= (\prod_i f^{\chi_i^{K_i}})^h \\ &= (\chi^K)^h = \chi^{Kh}.\end{aligned}$$

Consider next an arbitrary $L \leq H$, $L \in \mathcal{I}$, where again χ is an even gate and χ^L commutes. Let $K \leq L, K \in \mathcal{I}$ be any sparse enough subgroup such that χ^K is generated by $f^{\hat{\mathfrak{L}}_\mathcal{I}}$. Then by the two previous paragraphs, χ^{Kh} can also be built, where h runs over right coset representatives for K in L. By Lemma 12, we can build χ^L. (To recall the argument, since χ^K commutes, the composition of the χ^{Kh} over coset representatives is precisely χ^L.) We conclude that $\langle f^{\hat{\mathfrak{L}}_\mathcal{I}} \rangle$ contains every gate lattice χ^{Lh} where χ is even, $L \leq H$ and $h \in H$ is arbitrary.

Next, consider an arbitrary commuting χ^L for $L \in \mathcal{I}$, where not necessarily $L \leq H$. By what we already showed, χ^K in $\langle f^{\hat{\mathfrak{L}}_\mathcal{I}} \rangle$ where $K \leq L \cap H, K \lhd G, K \in \mathcal{I}$ is arbitrary. Now conjugating the situation with some $g \in G$, we transform the original map f into some f^g, which is still a nontrivial homeomorphism that now commutes with H^g, and applying the entire discussion to it (observing $K \leq H^g$ by $K \leq H$ and normality), conjugates of f^g by elements of $\hat{\mathfrak{L}}_\mathcal{I}$ also generate the same gate lattice χ^K.

For some χ_i, K_i we now have

$$\chi^K = \prod_i (f^g)^{\chi_i^{K_i}},$$

and we can continue with

$$\begin{aligned}\chi^K = \prod_i (f^g)^{\chi_i^{K_i}} &= \prod_i (\chi_i^{-1})^{K_i} \circ g^{-1} \circ f \circ g \circ \chi_i^{K_i} \\ &= \prod_i g^{-1} \circ ((\chi_i^{-1})^{K_i})^{g^{-1}} \circ f \circ (\chi_i^{K_i})^{g^{-1}} \circ g \\ &= (\prod_i (\chi_i^{-1})^{K_i g^{-1}} \circ f \circ \chi_i^{K_i g^{-1}})^g.\end{aligned}$$

Conjugating both sides by g^{-1}, we see that $\langle f^{\hat{\mathfrak{L}}_\mathcal{I}} \rangle$ contains the element $(\chi^K)^{g^{-1}} = \chi^{Kg^{-1}}$ for arbitrary $g \in G$. In particular by composing these with $g^{-1} = th$ where t ranges over right coset representatives of K in L, we obtain precisely χ^{Lh}. □

Theorem 2. *Let X be an EFP SFT on a residually finite group G. Then the group $\hat{\mathfrak{L}}_\mathcal{I}$ is simple.*

Proof. If G is finite, this is trivial as $\hat{\mathfrak{L}}_{\mathcal{I}}$ is just the alternating group. Otherwise, let $f \in \hat{\mathfrak{L}}_{\mathcal{I}}$ be arbitrary, i.e. f is a product of some elements $(\chi_i)^{H_i g_i}$ where the χ_i are even gates. Picking any normal subgroup $H \in \mathcal{I}$ of G contained in the intersection of the H_i, Lemma 12 shows that f is a product of gates of the form $(\chi_i)^{H g_i}$ with H normal. By Lemma 13, f is an automorphism for the H-action. By Lemma 28,

$$\langle f^{\hat{\mathfrak{L}}_{\mathcal{I}}} \rangle = \langle f, \hat{\mathfrak{L}}_{\mathcal{I}} \rangle$$

so the smallest normal subgroup of $\hat{\mathfrak{L}}_{\mathcal{I}}$ containing f is all of $\hat{\mathfrak{L}}_{\mathcal{I}}$. This concludes the proof of simplicity. □

Lemma 29. *Let X be a subshift on any residually finite group G. Then $\mathfrak{L}$ is a normal subgroup of the stabilized automorphism group.*

Proof. For finite G this is trivial, since $\mathfrak{L}$ is the entire symmetric group on X. It suffices to show $(\chi^{Kg})^f \in \mathfrak{L}$ whenever $f \in \mathrm{Aut}(X, H)$ for finite-index H and K is a sparse enough normal finite-index subgroup of H.[2] We calculate

$$\begin{aligned}
(\chi^{Kg})^f &= (\prod_{k \in K} \chi^{kg})^f \\
&= \prod_{k \in K} (\chi^{kg})^f \\
&= \prod_{k \in K} f^{-1}h^{-1}t^{-1}k^{-1}\chi kthf \\
&= \prod_{k \in K} h^{-1}f^{-1}t^{-1}k^{-1}\chi ktfh \\
&= \prod_{k \in K} h^{-1}t^{-1}f'^{-1}k^{-1}\chi kf'th \\
&= \prod_{k \in K} h^{-1}t^{-1}k^{-1}f'^{-1}\chi f'kth \\
&= (\chi^{f'})^{Kg}
\end{aligned}$$

Here, the second inequality follows from Lemma 5 and commutation of $(\chi^{kg})^f$ for different $k \in K$, where the latter in turn follows from the Curtis-Hedlund-Lyndon theorem (similarly as in the proof of Lemma 7) and sufficient sparseness of K. The third equality follows by opening up the conjugations and letting $g = th$ where $h \in H$ and t is one of finitely many left coset representatives for H. The fourth uses $f^h = f$. The fifth defines $f' = f^{t^{-1}}$. The fifth uses that f' is an automorphism of $(X, H^{t^{-1}})$, and $K \leq H^{t^{-1}}$ because K it is normal in G and $K \leq H$. The last step rewraps the definition; here for the commutation of the product we note that f' is one of $[G : H]$ many automorphisms of subshifts, namely subshifts $(X, H^{t^{-1}})$, thus satisfies the Curtis-Hedlund-Lyndon theorem; we simply initially take K sparse enough for these finitely many maps.

We have shown that conjugating even gate lattices by $f \in \mathrm{Aut}(X, H)$ produces gate lattices, and by the assumption $\hat{\mathfrak{L}} = \mathfrak{L}$ we conclude that $\hat{\mathfrak{L}}$ is normal. □

[2]The fact we have to refine with H here is the reason why $\mathfrak{L}_{\mathcal{I}}$ is not obviously normal for general $\mathcal{I}$ – of course if it is normal then by simplicity of $\mathfrak{L}$ we have $\mathfrak{L} = \mathfrak{L}_{\mathcal{I}}$.

Example 2: Even if χ is even, the $\chi^{f'}$ appearing in the above formula need not be even, or at least one needs to be careful about the choice of the strong neighborhood. Namely, pick $G = \mathbb{Z}$, and consider χ and χ^σ. Of course, they have the same parity if computed with different strong neighborhoods (and therefore with any large enough strong neighborhood both are even), but we show that it is sometimes possible to find a strong neighborhood that works for both, but is even for exactly one of them.

For this, pick $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and let X be the corresponding edge shift, i.e. at each $n \in \mathbb{Z} + \frac{1}{2}$ we have a vertex, and at each $n \in \mathbb{Z}$ we have an edge. Let us call the vertices $\{a, b\}$, and note that M simply says we have a unique edges between each pair of vertices. Now pick χ the unique gate with strong neighborhood $[0, 1]$ (so χ sees two edges) that permutes the word nontrivially if and only if the vertex at $-\frac{1}{2}$ is a. In cycle notation, the action on vertices at $(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ is

$$(aaa\ aba)(aab\ abb)(baa)(bab)(bba)(bbb).$$

Note that χ^σ performs the same modification on the word (consisting of edges) appearing in $[1, 2]$. Now consider these with strong neighborhood $[0, 2]$. Obviously χ is now an even gate, as in any context where the vertex at $-\frac{1}{2}$ is a we perform two swaps, and in any context where it is b, we do nothing. On the other hand, χ^σ always performs an odd permutation, since to get a nontrivial action we must pick the vertex at $\frac{1}{2}$ to be a. ○

Theorem 2, combined with the above lemma, immediately gives the following.

Theorem 3. *Let X be an EFP SFT on a residually finite group G. If $\hat{\mathfrak{L}} = \mathfrak{L}$, it is a maximal simple subgroup of the stabilized automorphism group of X.*

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