Color Blind Cellular Automata

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We introduce the classes of color blind and typhlotic cellular automata, that is, cellular automata that commute with all symbol permutations and all symbol mappings, respectively. We show that color blind cellular automata form a relatively large subclass of all cellular automata which contains an intrinsically universal automaton. On the other hand, we give simple characterizations for the color blind CA which are also group homomorphisms, and for general typhlotic CA, showing that both must be trivial in most cases. Using an extension lemma for a generalization of color blind CA, we show that the centralizer of a finite equicontinuous family of CA always contains an intrinsically universal automaton, as does that of an almost equicontinuous family with a blocking word of certain special type.

Key words: cellular automaton, commutation, symbol permutation, homomorphism, intrinsic universality, equicontinuity

1 INTRODUCTION

Suppose we wish to study a cellular automaton f, that is, a continuous shift invariant function from $S^{\mathbb{Z}}$ to itself, where S is a finite alphabet. A natural direction of study would be to consider its relation to some other cellular automata, for example to find the (group or monoid theoretic) *centralizer* of f, the set of all cellular automata it commutes with. This is known as the *commuting block maps problem*, and it has a long history, dating back to the 70s [3]. Algebraically, the centralizer of f can also be viewed as the set of homomorphisms of the unary algebra $(S^{\mathbb{Z}}, f)$ that are also cellular automata. To generalize this notion, one defines the centralizer of a whole family of cellular automata as the set of those CA that commute with all of them.

In this article, we study so-called color blind (typhlotic) cellular automata, that is, automata which commute with all symbol permutations (all symbol mappings, respectively), on full shifts and their subshifts. In other words, color blind cellular automata form the centralizer of the family of all cell-wise permutations. They are interesting mainly from the mathematical point of view, as the centralizer of a simple but nontrivial class of CA. We give a natural logical characterization of color blind cellular automata and show that there exists an intrinsically universal color blind CA. We also prove an extension lemma for a generalization of this concept, where we only consider a subset of all symbol permutations, and possibly proper subshifts of the full shift. Perhaps somewhat surprisingly, we show that intrinsic universality is also possible in typhotic CA if the full shift is binary, but that every typhlotic CA must be a shift map on other full shifts. We also show that in a quantitative sense, the class of color blind cellular automata is relatively as large as possible in the class of all cellular automata.

We then consider the case of full shifts over a finite group alphabet. The natural self-maps of such objects are the cellular automata that are also group homomorphisms for the product group structure, and we call them homomorphic cellular automata. We investigate cellular automata that are both color blind and homomorphic. This turns out to be very restrictive, and the situation is similar to that of typhlotic CA without the group structure: if the alphabet group is sufficiently simple ($\mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_2^2$), then there exist nontrivial color blind homomorphic CA, but on more complicated full group shifts, all color blind homomorphic CA are shift maps.

In the final section, we move to a slightly more general framework, and consider finite equicontinuous families of reversible cellular automata, of which sets of symbol permutations are an example. We show that the centralizer of such a family always contains an intrinsically universal automaton, using a simple recoding argument and the aforementioned extension lemma. We also briefly discuss almost equicontinuous families of cellular automata.

2 DEFINITIONS

Let S be a finite set, called the *alphabet*. The *full shift* is the space $S^{\mathbb{Z}}$ of infinite configurations over S endowed with the product topology. The topology can also be defined by the metric $d(x, y) = \inf\{2^{-n} \mid n \in \mathbb{N}, x_{[-n,n]} =$

$y_{[-n,n]}\}$. For $x \in S^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we denote by x_n the symbol of x at coordinate n. For a word $w \in S^*$ and $s \in S$, we denote by $|w|$ the length of w, and by $|w|_s$ the number of occurrences of s in w. For a configuration $x \in S^{\mathbb{Z}}$, we say w *occurs in* x if $w = x_{[n,n+|w|-1]}$ for some $n \in \mathbb{Z}$.

A subset $X \subset S^{\mathbb{Z}}$ is called a *subshift* if it is closed in the topology and invariant under the *shift map* $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$, defined by $\sigma(x)_n = x_{n+1}$. Alternatively, a subshift X is defined by a set $\mathcal{F} \subset S^*$ of *forbidden words* as the set of configurations in which no $w \in \mathcal{F}$ occurs. If $\mathcal{F}$ can be taken finite, X is a *subshift of finite type* (SFT for short). A configuration $x \in S^{\mathbb{Z}}$ is *spatially periodic* if $\sigma^p(x) = x$ for some $p > 0$.

A continuous mapping $f : X \to X$ in a subshift that commutes with σ is called a *cellular automaton*. All cellular automata f are defined by *local functions* $F : S^N \to S$, where $N \subset \mathbb{Z}$ is the finite *neighborhood* of f, by the formula $f(x)_n = F(x_{n+N})$ for all $n \in \mathbb{Z}$ [6]. We usually define $N = [-r, r]$ for some $r \in \mathbb{N}$, called the *radius* of f. To each CA f we associate a local function f_{loc}, which in general is not uniquely defined, but this should not cause any confusion. A configuration $x \in X$ is called *temporally periodic (with respect to f)* if $f^p(x) = x$ for some $p > 0$. A symbol mapping $\pi : S \to S$ can also be seen as a cellular automaton on $S^{\mathbb{Z}}$ by $\pi(x)_n = \pi(x_n)$.

For a set F of functions from a set X to itself, we denote by F^* the closure of F under composition, which also contains the identity map.

Let S be a finite algebra. Then, $S^{\mathbb{Z}}$ becomes an algebra when the operations are applied cellwise. We say a CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is *homomorphic* (with respect to the algebraic structure of S) if it is a homomorphism of $S^{\mathbb{Z}}$. This is the case exactly when the local rule $F : S^N \to S$ is a homomorphism. If S is an abelian group and f is of the form $\sum_{i=0}^{k-1} \sigma^{n_i}$ for some $n_i \in \mathbb{Z}$, we say f is a *sum of shifts*, and if the n_i are pairwise distinct, f is a *sum of distinct shifts*.

Let X be a nonempty set, and let $Q \subset 2^X$. If $\emptyset \notin Q$ and for all $A, B \in Q$ and $C \subset X$ we have $A \subset C \implies C \in Q$ and $A \cap B \in Q$, then Q is a *filter* on X. If Q is a maximal filter on X with respect to inclusion, then it is an *ultrafilter*.

Remark 1. *In the literature, the terminology related to cellular automata that are also group homomorphisms varies wildly. For example, in [10], the authors use the terms additive CA and group CA for cellular automata that are homomorphic with respect to an abelian group alphabet, and the term k-rule for CA that are sums of k distinct shifts. In [11] the term linear CA is used for homomorphic cellular automata. On the other hand, in [9] and* *many subsequent articles, the term linear CA refers to cellular automata on $\mathbb{Z}_p^{\mathbb{Z}}$ that we would call sums of shifts. Of course, over the alphabet $\mathbb{Z}_p$ the notions of homomorphic CA and sum of shifts coincide, but not over general abelian group alphabets. Even worse, the term linear is sometimes used to refer to* one-dimensional *CA. We have chosen our terminology in the hope of being as unambiguous as possible.*

3 COLOR BLIND CELLULAR AUTOMATA

We begin with the definition of our objects of interest, the color blind cellular automata. The definition is more general than this, mainly because of the applications in Section 7.

Definition 1. *Let $\Pi \subset S^S$ be a set of functions from S to itself. A cellular automaton $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ satisfying $\pi \circ f = f \circ \pi$ for all $\pi \in \Pi$ is called* Π-blind*. If Π is the set of bijections from S to itself, we say f is* color blind*, and if $\Pi = S^S$, we say f is* typhlotic.

In other words, the set of Π-blind cellular automata on $S^{\mathbb{Z}}$ is exactly the centralizer of Π in the monoid of all cellular automata on $S^{\mathbb{Z}}$ with respect to composition. Another way to express this is that the set of spacetime diagrams of a Π-blind CA is closed under cellwise applications of elements of Π, and in particular, in a spacetime diagram of a color blind cellular automaton, the colors can be renamed in any way. We use the somewhat obscure term typhlotic, meaning blind, to avoid cluttering the global namespace of cellular automata: we will soon see that these automata are rather trivial (Theorem 2), and presumably do not have much theory beyond what we prove in this article.

Example 1. *The radius-1 cellular automaton f on $\{0, 1, 2\}^{\mathbb{Z}}$ defined by*

$$f_{\mathrm{loc}}(a, b, c) = \begin{cases} c, & \text{if } a = b \neq c, \\ a, & \text{if } a \neq b = c, \\ b, & \text{otherwise} \end{cases}$$

is clearly color blind. It always chooses the symbol in its neighborhood that is in the minority, or acts as the identity CA if such a symbol does not exist. A portion of a spacetime diagram of f is shown in Figure 1.

A cellular automaton f on $S^{\mathbb{Z}}$ is called *captive* if the local rule f_{loc} satisfies $f_{\mathrm{loc}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for all $a_1, \ldots, a_n \in S$. Color blind CA are 'almost captive' in the following sense.

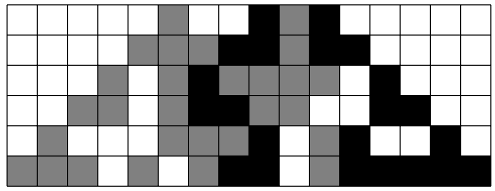

FIGURE 1
A sample spacetime diagram of the cellular automaton of Example 1, with time advancing downward. The labels of the colors are unimportant, since any symbol permutation of a spacetime diagram of a color blind CA is also its spacetime diagram.

Lemma 1. *Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a color blind CA. Then $f_{\mathrm{loc}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ whenever $|\{a_1, \ldots, a_n\}| < |S| - 1$.*

Proof. Suppose for a contradiction that we have $|\{a_1, \ldots, a_n\}| < |S| - 1$, but $a = f_{\mathrm{loc}}(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. Then, there exists

$$b \in S \setminus \{a, a_1, \ldots, a_n\}.$$

Now, f does not commute with the transposition $(a\ b)$. □

The automaton of Example 1 is captive by definition. However, not all color blind automata are captive, since the local rule may output the 'last remaining color' unambiguously when all but one color appear in the neighborhood, as the alphabet size is known. The following is an example of this phenomenon.

Example 2. *The radius-1 cellular automaton f on $\{0, 1, 2\}^{\mathbb{Z}}$ defined by*

$$f_{\mathrm{loc}}(a, b, c) = \begin{cases} d, & \text{if } |\{a, b, c\}| = 2 \text{ and } d \notin \{a, b, c\}, \\ b, & \text{otherwise} \end{cases}$$

is color blind. It always chooses the unique symbol that does not appear in its neighborhood, or acts as the identity CA if such a symbol does not exist. It is clearly not captive. A portion of a spacetime diagram of f is shown in Figure 2.

Typhlotic CA are in fact captive, which we will obtain as a corollary of Theorem 2. We continue with a simple logical characterization of color blind cellular automata which motivates their definition.

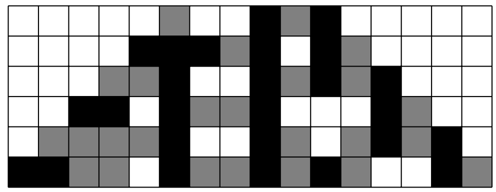

FIGURE 2
A sample spacetime diagram of the cellular automaton of Example 2, with time advancing downward.

Definition 2. *Fix a set of variables $V = \{v_1, \dots, v_n\}$. A* color blind equation *over V is a boolean combination of basic equations of the form $v_i = v_j$. For a symbol equation E over V, an alphabet S and a word $w \in S^n$, we denote by $E(w)$ the equation obtained by replacing each v_i by w_i in E. The equation E defines a set of words $E(S) \subset S^n$ by $E(S) = \{w \in S^n \mid E(w) \text{ holds}\}$. We say E is* captive on *S if the last letter of w occurs at least twice in w for all $w \in E(S)$, and* captive*, if it is captive on S for all finite S. If $n = 2r + 2$ and $E(S)$ defines a function from S^{2r+1} to S (seen as a subset of $S^{2r+1} \times S$), we let $f_E^S : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be the cellular automaton whose local function it is. We say f_E^S is* defined by a color blind equation.

Lemma 2. *A set of words $W \subset S^n$ is defined by a color blind equation if and only if it is closed under symbol permutations.*

Proof. First, let $W = E(S)$ for an equation E, and consider an arbitrary symbol permutation $\pi : S \to S$. It is clear that if $E(w)$ holds for a word $w \in S^n$, then so does $E(\pi(w))$, and thus W is closed under symbol permutations.

Suppose then that W is closed under symbol permutations. For all $w \in W$, define the equation $E_w = \bigwedge_{i,j \in [0,n-1]} t(i,j)$, where $t(i,j)$ is $v_i = v_j$ if $w_i = w_j$, and $\neg(v_i = v_j)$ otherwise. We let $E = \bigvee_{w \in W} E_w$. Now, it is clear that $W \subset E(S)$. On the other hand, let $v \in E(S)$. This means that $v \in E_w(S)$ for some $w \in W$. It is easy to see that there then exists a symbol permutation $\pi : S \to S$ with $\pi(w) = v$, and since W is closed under symbol permutations, we have $v \in W$. $\square$

As a cellular automaton commutes with symbol permutations if and only if its local rule does, we obtain the following corollary.

Corollary 1. *A CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is (captive and) color blind if and only if it is defined by a (captive and) color blind equation.*

Example 3. *The cellular automaton of Example 1 is defined by the captive and color blind equation*

$$(v_1 = v_2 \wedge v_2 \neq v_3 \wedge v_3 = v_4) \vee (v_1 \neq v_2 \wedge v_2 = v_3 \wedge v_1 = v_4) \vee (v_2 = v_4),$$

where v_1, v_2, v_3 and v_4 correspond to a, b, c and $f_{\mathrm{loc}}(a, b, c)$ in the definition, respectively.

The characterization essentially says that a cellular automaton is color blind if and only if it can be defined without referring to any particular colors, but only their arrangements on the neighborhood. If we restrict to captive color blind cellular automata, the equation defining the color blind CA can be chosen so that it defines a CA on any full shift containing the original. Proposition 1 generalizes this idea, and to prove it, we need a few definitions and a standard topological lemma.

Lemma 3 (Pasting Lemma)**.** *Let X and y be topological spaces, let $A, B \subset X$ be closed, and let $f : A \cup B \to Y$ be a function. If $f|_A$ and $f|_B$ are continuous, then so is f.*

Definition 3. *Let $X \subset S^{\mathbb{Z}}$ be a subshift, let $f : X \to X$ be a CA, and let $\Pi \subset S^S$. Suppose that whenever $\pi \in \Pi^*$ and $x \in X$ satisfy $\pi(x) \in X$, then $\pi(f(x)) = f(\pi(x))$. Then we say f is* Π-blind on X*. If there exists $r \in \mathbb{N}$ such that for all $x \in X$ there exists $k \in [-r, r]$ such that $f(x) = x_k$, we say f is* captive on X.

The definition of Π-blindness on a subshift leads to some unintuitive situations, as shown in the following.

Example 4. *Let $S = \{0, 1, 2\}$ and $X = \{0, 1\}^{\mathbb{Z}}$. Then the symbol permutation $f = (0\ 1)$ is* not *color blind on X with respect to all symbol permutations of S. Namely, if we denote $\pi = (1\ 2)$, then $\pi({}^{\infty}0^{\infty}) = {}^{\infty}0^{\infty} \in X$, but $f(\pi({}^{\infty}0^{\infty})) = {}^{\infty}1^{\infty} \neq {}^{\infty}2^{\infty} = \pi(f({}^{\infty}0^{\infty}))$.*

However, it is exactly the right definition if we want it to correspond to f being a restriction of a Π-blind cellular automaton on $S^{\mathbb{Z}}$. The following extension result shows this and more.

Proposition 1. *Let $X \subset S^{\mathbb{Z}}$ be a subshift, and let Π be a set of permutations on S. Then a CA $f : X \to X$ is (captive and) Π-blind on X if and only if $f = g|_X$ for a (captive and) Π-blind CA $g : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ which satisfies $g(X) \subset X$.*

Proof. First, if $f = g|_X$ for such a $g : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$, and $\pi \in \Pi^*$ is arbitrary, then $\pi(g(x)) = g(\pi(x))$ for all $x \in S^{\mathbb{Z}}$, so in particular this is the case when $x, \pi(x) \in X$, and then $\pi(f(x)) = f(\pi(x))$. In this case f is also clearly captive on X, if g is captive.

For the other direction, we first claim that X can be assumed to be closed under the action of Π^*. Namely, replace X by $\hat{X} = \bigcup_{\pi \in \Pi^*} \pi(X)$, which as a finite union of subshifts is a subshift, and for all $\pi \in \Pi^*$, define $f_\pi : \pi(X) \to \pi(X)$ by $f_\pi(\pi(x)) = \pi(f(x))$. Now, if $x \in \pi(X) \cup \rho(X)$, then $x = \pi(y) = \rho(z)$ for some $x, y \in X$. Since Π^* is a subgroup of the symmetric group on S, we have $\rho^{-1} \circ \pi \in \Pi^*$, and thus $\rho^{-1}(\pi(y)) = z \in X$. Since f is Π-blind, this implies $f(z) = f(\rho^{-1}(\pi(y))) = \rho^{-1}(\pi(f(y)))$, that is,

$$f_\rho(x) = f_\rho(\rho(z)) = \rho(f(z)) = \pi(f(y)) = f_\pi(\pi(y)) = f_\pi(x).$$

By the Pasting Lemma, the well-defined function $\hat{f} : \hat{X} \to \hat{X}$ such that $\hat{f}|_{\pi(X)} = f_\pi$ for all $\pi \in \Pi^*$ is continuous. It is also Π-blind by definition, and clearly captive if f is.

Suppose thus that X is closed under Π^*, and let $r \in \mathbb{N}$ be a radius for f that also witnesses its captivity, if f is captive. Define the radius-r CA $g : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ by the local rule

$$g_{\mathrm{loc}}(w) = \begin{cases} f_{\mathrm{loc}}(w), & \text{if } w \in \mathcal{B}_{2r+1}(X) \\ w_r, & \text{otherwise.} \end{cases}$$

It is clear that $g|_X = f$ (and thus $g(X) \subset X$) and that g is captive if f is. It remains to be shown that g is Π-blind, and for that, let $\pi \in \Pi^*$ and $x \in S^{\mathbb{Z}}$ be arbitrary, and let $i \in \mathbb{Z}$. We now have $x_{[i-r,i+r]} \in \mathcal{B}_{2r+1}$ if and only if $\pi(x_{[i-r,i+r]}) \in \mathcal{B}_{2r+1}$. In the former case,

$$g(\pi(x))_i = f_{\mathrm{loc}}(\pi(x_{[i-r,i+r]})) = \pi(f_{\mathrm{loc}}(x_{[i-r,i+r]})) = \pi(g(x))_i,$$

since X is closed under the action of π. In the latter case, we have $g(\pi(x))_i = \pi(x)_i = \pi(g(x))_i$. This shows that g is Π-blind. $\square$

From the above and Corollary 1 we deduce the following.

Corollary 2. *Let $X \subset S^{\mathbb{Z}}$ be a subshift. Then a CA $f : X \to X$ is (captive and) color blind on X if and only if $f = f_E^S|_X$ for a (captive) color blind equation E.*

There are color blind cellular automata defined on strict subshifts of the full shift which are not captive, and the symbol permutation $(0\ 1)$ on $X =$

$\{^\infty 0^\infty, {}^\infty 1^\infty\}$ is a simple example. However, the restriction to sets of symbol permutations is necessary, as shown by Theorem 2.

We also show by another example that the radius of f may not be sufficient to define g even if it witnesses the captivity of f on X. Let X consist of the configurations $x = {}^\infty(0122)^\infty$ and $y = {}^\infty(0022)^\infty$ and their shifts, and define $f_{\mathrm{loc}} : \{0,1,2\}^3 \to \{0,1,2\}$ by $f_{\mathrm{loc}}(a,b,c) = b$, except for $f_{\mathrm{loc}}(0,1,2) = 0$. Now, f is captive and color blind on X (the only symbol permutation we need to check is $(0\ 2)$). However, no color blind cellular automaton on $\{0,1,2\}^{\mathbb{Z}}$ with radius 1 is an extension of f, since $f_{\mathrm{loc}}(012) = 0 = f_{\mathrm{loc}}(201)$. One can check that the neighborhood $N = [-1,2]$ suffices though.

After Proposition 1 has been established, a natural question arises: if we have a sufficiently 'well-behaved' subshift $Y \subset S^{\mathbb{Z}}$ that contains X, is it possible to extend f to a CA $g : Y \to Y$ which is Π-blind on Y? The following example shows that assuming Y to be a mixing nearest-neighbor SFT closed under Π is not sufficient.

Example 5. *Let $S = \{0,1,\#,a,b,c,d\}$ and $X = \{^\infty 0^\infty, {}^\infty 1^\infty\}$, and define the mixing SFT $Y \subset S^{\mathbb{Z}}$ by the allowed 2-words*

$$\{00, 11, \#\#, 0a, 0b, 1c, 1d, a\#, b\#, c\#, d\#, \#0, \#1\}.$$

Define the symbol permutations $\pi = (a\ b)$ and $\rho = (c\ d)$ and let $\Pi = \{\pi, \rho\}$, so that both X and Y are closed under Π, and the automaton $f : X \to X$ that swaps ${}^\infty 0^\infty$ and ${}^\infty 1^\infty$ is Π-blind. We claim that f cannot be extended to a Π-blind automaton $g : Y \to Y$. Assume the contrary, and consider the configuration $y = {}^\infty 0a\#^\infty \in Y$. Since $g|_X = f$, the image $g(y) \in Y$ is left asymptotic to ${}^\infty 1^\infty$. Now, if $g(y) \neq {}^\infty 1^\infty$, then $g(y)_i \in \{c,d\}$ for some $i \in \mathbb{Z}$, but then $\rho(g(y)) \neq g(y)$, even though $\rho(y) = y$, contradicting the Π-blindness of g. Thus $g(y) = {}^\infty 1^\infty$, implying $g(^\infty\#^\infty) = {}^\infty 1^\infty$. Similarly we see that $g(^\infty 1c\#^\infty) = {}^\infty 0^\infty$, implying $g(^\infty\#^\infty) = {}^\infty 0^\infty$, a contradiction.

We are not aware of an easy characterization for the situations where the extension succeeds, but there is at least the following sufficient condition. We present an application for it in Section 7.

Proposition 2. *Let $X \subset Y \subset S^{\mathbb{Z}}$ be a subshift, of which Y is a mixing SFT, and let Π be a set of permutations on S such that Y is closed under Π. Suppose further that the only element of Π^* having a fixed point in X is the identity. Then a CA $f : X \to X$ is (captive and) Π-blind on X if and only if $f = g|_X$ for a CA $g : Y \to Y$ which is (captive and) Π-blind on Y and satisfies $g(X) \subset X$.*

Proof. The backward implication is proved as in Proposition 1. For the forward implication, we can again assume that X is closed under Π^*.

Let $r \in \mathbb{N}$ be a radius for f, a window size and mixing distance for Y, and such that the only element of Π^* having a fixed point in $\mathcal{B}_{2r}(X)$ is the identity. We now define the automaton g, and for that, let $y \in Y$. We say that a coordinate $i \in \mathbb{Z}$ is *y-good*, if $y_{[i-r,i+r]} \in \mathcal{B}(X)$, and a *$y$-good block* is an interval $[i, i+k-1] \subset \mathbb{Z}$ of length $k \geq 2r$ such that $[i-r, i+k+r-1]$ contains only y-good coordinates. A coordinate is *y-excellent*, if it is contained in some y-good block. Then, a *y-bad block* is an interval $[i, i+k-1] \subset \mathbb{Z}$ of length $k \geq r$ such that $[i-r, i+k+r-1]$ contains no y-excellent coordinates. A coordinate is *y-dangerous* if it is contained in some y-bad block. It is easy to see that the coordinates that are neither y-excellent nor y-dangerous (we call them *y-transitional*) form intervals of length at least r and at most $3r-1$ that are bordered by y-good blocks at least on one side.

If a coordinate $i \in \mathbb{Z}$ is y-excellent, we define $g(y)_i = f_{\text{loc}}(y_{[i-r,i+r]})$, and if it is y-dangerous, we define $g(y)_i = y_i$. If $I \subset \mathbb{Z}$ is a y-good interval, we then have $g(y)_I \in \mathcal{B}(X)$. Suppose then that $[i, i+k-1]$ is a maximal interval of y-transitional coordinates, so that $r \leq k < 3r$. Without loss of generality we assume that the coordinate $i-1$ is y-excellent. Let $uvw = y_{[i-2r,i+k+2r-1]}$, where $|u| = |w| = 2r$, so that $u \in \mathcal{B}_{2r}(X)$. Denote also $h(u,v) = g(y)_{[i-r,i-1]} \in \mathcal{B}_r(X)$, and note that h commutes with the elements of Π^*. Consider the finite set $\tau \subset \mathcal{B}_{2r}(X) \times \mathcal{B}(Y) \times \mathcal{B}(Y)$ of all the triples (u,v,w) obtained this way, and note that τ is closed under Π^*.

For every $t = (u,v,w) \in \tau$, let $\pi \in \Pi^*$ be such that $\pi(t)$ is the lexicographically largest element in the Π^*-orbit of t. Let $G(\pi(t)) \in \mathcal{B}_{|v|}(Y)$ be such that $h(\pi(u), \pi(v)) \cdot G(\pi(t)) \cdot \pi(w) \in \mathcal{B}(Y)$, and define $G(t) = \pi^{-1}(G(\pi(t)))$. If we have $\pi(t) = \rho(t)$ for some $\rho \in \Pi^*$, then $\pi(u) \in \mathcal{B}_{2r}(X)$ is a fixed point for $\pi^{-1} \circ \rho$, and thus $\rho = \pi$, which implies that $G(t)$ is well-defined. We have defined a Π^*-commuting function G from τ to $\mathcal{B}(Y)$ such that $|G(u,v,w)| = |v|$ and $h(u,v) \cdot G(t) \cdot w \in \mathcal{B}(Y)$, and if $[i, i+k-1]$ is a maximal interval of y-transitional coordinates, we define $g(y)_I = G(y_{[i-2r,i+k+r-1]})$. The g-images of those transitional intervals for which $i-1$ is y-dangerous are obtained symmetrically, since then $i+k+r$ is y-excellent.

We now claim that $g : Y \to Y$ is a cellular automaton that is Π-blind on Y (and captive on Y if f is captive on X) and satisfies $g|_X = f$. The claim about captivity is easy to see, and g is clearly shift-commuting and continuous. Also, the image of every $y \in Y$ is also in Y by the fact that r is a window size for Y, and we have $g|_X = f$ by construction, since for all

$x \in X$, every coordinate is x-excellent.

It remains to be shown that g is Π-blind on Y, so let $y \in Y$ and $\pi \in \Pi^*$. Since X, and thus $\mathcal{B}(X)$, is closed under Π^*, the excellent, dangerous and transitional coordinates are the same in y and $\pi(y)$. If $i \in \mathbb{Z}$ is dangerous, then $g(\pi(y))_i = \pi(y)_i = \pi(g(y))_i$, and if it is excellent, then $g(\pi(y))_i = f_{\mathrm{loc}}(\pi(y)_{[i-r,i+r]}) = \pi(f_{\mathrm{loc}}(y_{[i-r,i+r]})) = \pi(g(y))_i$ by the Π-blindness of f. Suppose thus that i is transitional, and thus contained in a maximal interval $I = [j, j+k-1]$ of transitional coordinates. We again assume that $j-1$ is excellent, so that $y_{[j-2r,j+k+r-1]} = t = (u, v, w) \in \tau$. Let $\rho(t) \in \tau$ be the lexicographically maximal element in the Π^*-orbit of t. Then we have $g(y)_I = \rho^{-1}(G(\rho(u), \rho(v)))$ and $g(\pi(y))_I = \pi(\rho^{-1}(G(\rho(u), \rho(v)))) = \pi(g(y))_I$. In particular $g(\pi(y))_i = \pi(g(y))_i$, and thus g is Π-blind on Y. $\square$

4 CONSTRUCTING COLOR BLIND CELLULAR AUTOMATA

In this section, we give concrete examples of color blind cellular automata, and prove some results that require explicit construction of such objects.

Definition 4. *Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a cellular automaton with neighborhood size n. We say f is a* majority CA *if, whenever $f_{\mathrm{loc}}(s_1, \ldots, s_n) = s$, we have $|\{i \in [1, n] \mid s_i = s\}| \geq |\{i \in [1, n] \mid s_i = s'\}|$ for all $s' \in S$.*

This means that the local rule of a majority CA always outputs a symbol that occurs a maximal number of times in the input. All majority CA are of course captive. In the binary case, there is a unique majority CA for each odd neighborhood size, and this CA is color blind. In other cases, the CA must have a tie-breaking rule. To make such a CA color blind we can, for example, always choose the leftmost input symbol s_m that maximizes $|\{i \in [1, n] \mid s_i = s_m\}|$.

Of the 256 elementary cellular automata (see [17] for the definitions and the numbering scheme), 16 rules are color blind. The even-numbered rules are summarized in Table 1, while the odd-numbered rules are obtained by subtracting their numbers from 255, effectively composing them with the symbol permutation $(0\ 1)$. We show the even-numbered color blind rules, as they are exactly the captive ones. Of these 8 elementary automata, the most interesting ones are 150 and 142. Rule 150 is a sum of three distinct shifts, and some properties of rule 142 are studied in at least [5].

In the next result, *intrinsic universality* is understood with respect to simulation by injective bulking. In this formalism, a cellular automaton $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ *simulates* another automaton $g : T^{\mathbb{Z}} \to T^{\mathbb{Z}}$ if there exists an injective

TABLE 1
The even-numbered color blind elementary CA. The variables v_1, v_2 and v_3 denote inputs to the local rule, and v_4 is its output.

CA	Color blind equation	Description
142	$(v_4 \neq v_2) \iff (v_1 = v_2 \neq v_3)$	Left shift with 'barriers'
150	$(v_1 = v_2) \iff (v_4 = v_3)$	Sum of neighborhood mod 2
170	$v_4 = v_3$	Left shift
178	$(v_4 = v_2) \iff (v_1 = v_2 = v_3)$	Flip unless all inputs equal
204	$v_4 = v_2$	Identity
212	$(v_4 \neq v_2) \iff (v_1 \neq v_2 = v_3)$	Mirrored 142
232	$(v_4 \neq v_2) \iff (v_1 \neq v_2 \neq v_3)$	Majority
240	$v_4 = v_1$	Right shift

function $\phi : T \to S^{m \times n}$ from T-symbols to S-rectangles such that for every spacetime diagram x of g, the configuration $\phi(x)$ is a spacetime diagram of f. An intrinsically universal automaton is then one that simulates any other CA. See [12] for the precise definitions; the main message of the theorem is that captive color blind cellular automata can be very complex both from the computational and the dynamical points of view.

Theorem 1. *For any alphabet S with $|S| \geq 2$, there exists an intrinsically universal captive color blind cellular automaton on $S^{\mathbb{Z}}$.*

Proof. It is enough to show that any single CA can be simulated by a captive color blind automaton, as there exists an intrinsically universal CA and injective simulations are composable. Let thus $g : [1, n-1]^{\mathbb{Z}} \to [1, n-1]^{\mathbb{Z}}$ be any cellular automaton, choose distinct symbols $a, b \in S$ and for all $i \in [1, n-1]$, let $w_i = a^i b^{2n-i}$. Define the injection $h : [1, n-1]^{\mathbb{Z}} \to S^{\mathbb{Z}}$ by

$$h(x) = w_{x_{-2}} w_{x_{-1}} . w_{x_0} w_{x_1} \cdots .$$

Let $Z = h([1, n-1]^{\mathbb{Z}})$ and $Y = \bigcup_{i=0}^{2n-1} \sigma^i(Z)$. It is easy to see that $Y \subset S^{\mathbb{Z}}$ is an SFT, since its configurations are exactly the infinite concatenations of the finitely many words $w_i \in a^+ b^+$. Define $f : Z \to Z$ by $f \circ h = h \circ g$. Now, f has a unique shift-commuting extension to a function $\hat{f} : Y \to Y$, which is then a cellular automaton simulating g. We may assume $\hat{f}$ has neighborhood $[-r, r]$ for some $r > 2n$. Then $\hat{f}$ is trivially captive and commutes with all symbol permutations of Y, since both symbols are always visible in the neighborhood, and no nontrivial symbol permutation keeps any configuration of Y inside it. Thus, $\hat{f}$ has a color blind extension to $S^{\mathbb{Z}}$ by Proposition 1. $\square$

In [2], the set $\mathrm{Aut}(X)$ of bijective cellular automata on a mixing SFT $X \subset S^{\mathbb{Z}}$ is considered (see the article for the precise definition). The *symmetry of* X is defined as the relative asymptotic density of $\mathrm{Aut}(X)$ in the set of all cellular automata on X: $s(X) = \limsup_{n\to\infty} \frac{1}{n} \log\log|\mathrm{Aut}(X)_n|$, where $\mathrm{Aut}(X)_n$ denotes the set of bijective cellular automata on X that can be defined on the neighborhood $[-\lfloor n/2\rfloor, \lceil n/2\rceil]$. Inspired by this, we define the following.

Definition 5. *Let $\mathcal{C}$ be a family of cellular automata on $S^{\mathbb{Z}}$. The* density *of $\mathcal{C}$ is defined as*

$$d(\mathcal{C}) = \limsup_{n\to\infty} \frac{1}{n} \log_{|S|} \log_{|S|} |\mathcal{C}_n|, \tag{1}$$

where $\mathcal{C}_n$ denotes the set of cellular automata in $\mathcal{C}$ that can be defined on the neighborhood $[-\lfloor n/2\rfloor, \lceil n/2\rceil]$.

We now show that color blind cellular automata are abundant in the sense of the previous definition. Note that the set $\mathcal{CA}$ of all cellular automata on $S^{\mathbb{Z}}$ has density 1, as $|\mathcal{CA}_n| = |S|^{|S|^n}$ for all $n \in \mathbb{N}$.

Proposition 3. *Denote by $\mathcal{CB}$ the set of captive color blind cellular automata on $S^{\mathbb{Z}}$. Then $d(\mathcal{CB}) = 1$.*

Proof. Let $S = \{s_1, \ldots, s_{|S|}\}$, and let $n \in \mathbb{N}$ be arbitrary. We define an injective map $\phi : \mathcal{CA}_n \to \mathcal{CB}_{n+|S|}$, which shows that $|\mathcal{CA}_n| \leq |\mathcal{CB}_{n+|S|}|$. For that, let $f \in \mathcal{CA}_n$ have neighborhood size n. The local function $\phi(f)_{\mathrm{loc}} : S^{n+|S|} \to S$ works as follows on the inputs $a_1, \ldots, a_{n+|S|} \in S$. If the symbols $a_{n+1}, \ldots, a_{n+|S|}$ are pairwise distinct, we let $\pi : S \to S$ be the symbol permutation that maps each a_{n+i} to s_i. The local function then returns $\pi^{-1}(f_{\mathrm{loc}}(\pi(a_1), \ldots, \pi(a_n)))$. If the symbols $a_{n+1}, \ldots, a_{n+|S|}$ are not pairwise distinct, $\phi(f)_{\mathrm{loc}}$ returns a_1. Then $\phi(f)$ is captive and color blind, and ϕ is injective.

Now, we calculate

$$\begin{aligned} &\frac{1}{n+|S|} \log_{|S|} \log_{|S|} |\mathcal{CB}_{n+|S|}| \geq \frac{1}{n+|S|} \log_{|S|} \log_{|S|} |\mathcal{CA}_n| \\ &= \frac{1}{n+|S|} \log_{|S|} \log_{|S|} |S|^{|S|^n} = \frac{n}{n+|S|} \overset{n\to\infty}{\longrightarrow} 1, \end{aligned}$$

which proves the claim. $\square$

We remark here that our definition of density measures the asymptotic growth rate of a set of cellular automata on a given alphabet, when the radius increases. An alternative perspective is taken in [16], where the radius $r \in \mathbb{N}$ is fixed, and the density of a set $\mathcal{C}$ is cellular automata is defined as the limit of $|\mathcal{C}^n|/|\mathcal{CA}^n|$, when it exists, where $\mathcal{C}^n$ is the set of CA in $\mathcal{C}$ with radius r on an alphabet of size n. Interestingly, it is shown in particular that the density of the set of all captive cellular automata is 0, so the *opposite* of the analogue of Proposition 3 holds in this formalism.

5 TYPHLOTIC CELLULAR AUTOMATA

We now turn our attention to typhlotic cellular automata, and start with the observation that they are not necessarily trivial. For example, the intrinsically universal CA given in Theorem 1 is in fact typhlotic in the case $|S| = 2$. Furthermore, every binary majority CA is typhlotic. These CA are already color blind, so we only need to check that they commute with the symbol maps that are not permutations, namely the constant maps $s \mapsto 0$ and $s \mapsto 1$. But this easily follows from the fact that both the intrinsically universal CA and majority CA are captive.

Somewhat curiously, if the alphabet S has more than two elements, the situation changes drastically. For example, as a corollary of Theorem 2, a ternary color blind majority CA can not be typhlotic unless it has a neighborhood of size 1. The proof of Theorem 2 follows from some rather general set theory. Namely, we show that a typhlotic CA is defined by an ultrafilter on its neighborhood, and ultrafilters on finite sets are very simple. We note that we do not need any hard set theoretic results on ultrafilters: they just happen to provide convenient terminology for the proof.

We start with two characterizations of ultrafilters. The first one is just the observation that the well-known partition property of ultrafilters characterizes them, as also the filter axioms follow from it. This result has already appeared in at least [7]. The second one is rather specific to typhloticity, and is in fact just the first part of Theorem 2 in thin disguise.

Lemma 4 (Corollary 1.6 of [7])**.** *Let X be a nonempty set, let $k \in \mathbb{N}$ with $k \geq 3$, and let $Q \subset 2^X$ have the property that for all partitions $(A_1, \ldots, A_k)$ of X, exactly one A_i is in Q. Then Q is an ultrafilter. Furthermore, every ultrafilter satisfies the property for every $k \geq 1$.*

Proof. First, from the partition $(X, \emptyset, \ldots, \emptyset)$ we deduce that $\emptyset \notin Q$ and $X \in Q$. Now, Q cannot contain two disjoint subsets $A, B \subset X$, as otherwise the partition $(A, B, X \setminus (A \cup B), \emptyset, \ldots, \emptyset)$ would contradict the assumptions. Thus, if $A \subset X$, then exactly one of A and $X \setminus A$ is in Q, by the partition $(A, X \setminus A, \emptyset, \ldots, \emptyset)$.

Suppose then that $A \in Q$ and $A \subset B$. The partition $(X \setminus B, A, B \setminus A, \emptyset, \ldots, \emptyset)$ proves that $X \setminus B \notin Q$, so by the above $B \in Q$. Finally, if $A, B \in Q$, then neither of $A \setminus B$ or $B \setminus A$ can be in Q, and then the partition $(A \setminus B, B \setminus A, A \cap B, \emptyset, \ldots, \emptyset)$ shows that $A \cap B \in Q$.

The converse claim is a well known property of ultrafilters. □

For the next lemma, we define a more general definition of typhloticity.

Definition 6. *Let S and T be sets with S finite, and let $f : S^T \to S$ be a function. Then we say f is* typhlotic *if for every function $g : S \to S$, we have $f \circ g = g \circ f$, where g is applied coordinatewise on the left side of the equation.*

Lemma 5. *Let T be a set, and S a finite set with $|S| \geq 3$. Then the map $f \mapsto \{\{i \in T \mid x_i = f(x)\} \mid x \in S^T\}$ is a bijection from the set of typhlotic maps $f : S^T \to S$ to the set of ultrafilters on T.*

Proof. Without loss of generality, let $S = [1, k]$. For all $x \in S^T$ and $s \in S$ we define $x|_s = \{i \in T \mid x_i = s\}$.

Let first $f : S^T \to S$ be typhlotic, and denote by $Q \subset 2^T$ the image of f under the mapping. By Lemma 4, we need to show that if (A_1, A_2, A_3) is a partition of T, then exactly one of the A_i is in Q, that is, of the form $x|_{f(x)}$ for some $x \in S^T$. First, since $k \geq 3$, there exists $x \in S^T$ such that $x|_i = A_i$ for all $i \in \{1, 2, 3\}$. Then $f(x) \in \{1, 2, 3\}$, for otherwise, letting $\pi : S \to S$ be the symbol map that sends $f(x)$ to 1 and otherwise acts as the identity, we would have $1 = \pi(f(x)) = f(\pi(x)) = f(x)$, a contradiction. Thus at least one of the A_i is in Q.

Suppose then that, for example, $A_1 = x|_{f(x)}$ and $A_2 = y|_{f(y)}$ for some $x, y \in S^T$, where we may assume $f(x) = 1$ and $f(y) = 2$ by applying symbol permutations. Let $z \in S^T$ be defined by $z|_i = A_i$ for all $i \in \{1, 2, 3\}$. If $f(z) = 1$, define the symbol map $\pi : S \to S$ by $\pi(2) = 2$ and $\pi(s) = 3$ for all $s \in S \setminus \{2\}$. Then

$$3 = \pi(f(z)) = f(\pi(z)) = f(\pi(y)) = \pi(f(y)) = \pi(2),$$

a contradiction. A symmetric argument shows that $f(z) \neq 1$ is likewise impossible. Thus exactly one of the A_i is in Q, and Q is a ultrafilter.

Conversely, let Q be an ultrafilter on T, and define $f : S^T \to S$ by $f(x) = a$ iff $\{i \in T \mid x_i = a\} \in Q$. Again by Lemma 4 (the converse direction), f

is then well-defined. Since ultrafilters are closed under supersets, f is easily seen to be typhlotic. As the ultrafilter corresponding to f is Q, this concludes the claim. $\square$

The following is also a well known property of ultrafilters (for instance, it appears as Example 1.3 in [7]).

Lemma 6. *Let T be finite and let Q be an ultrafilter on T. Then Q is principal, that is, $Q = \{A \subset T \mid j \in A\}$ for some $j \in T$.*

Proof. Since T is finite, we can take a minimal set A in Q. If A is a singleton, we are done. If A is not a singleton, Q is not a maximal filter. $\square$

Theorem 2. *If $|S| \geq 3$, the typhlotic CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ are exactly the shift maps. If $|S| = 2$, they are exactly the captive color blind CA.*

Proof. First, suppose $|S| \geq 3$, and let $N \subset \mathbb{Z}$ be the neighborhood of f. The local rule $f_{\mathrm{loc}} : S^N \to S$ is typhlotic since f is. Let Q be the ultrafilter on N that defines it, given by Lemma 5. Since N is finite, $Q = \{A \subset N \mid j \in A\}$ for some $j \in N$ by Lemma 6, which means

$$f(x)_0 = a \iff \{i \in N \mid x_i = a\} \in Q \iff x_j = a.$$

Thus f is a shift map.

In the case $|S| = 2$, a CA is captive if and only if it commutes with constant maps, and all symbol maps are either permutations or constant maps. This concludes the proof. $\square$

6 HOMOMORPHIC COLOR BLIND AUTOMATA

In Section 4, we saw that color blind cellular automata can do almost anything a general cellular automaton can do, with any alphabet size. On the other hand, typhlotic cellular automata turned out to be almost the same objects as color blind CA in the binary case, but shift maps for larger alphabets. In this section, we show that cellular automata that are color blind and *homomorphic* satisfy a similar property: if the group is very simple, the color blind homomorphic CA form a large subclass of all homomorphic CA, but when the group is larger, they are all shift maps.

Color blindness of homomorphic CA was also studied in [10], and there, the term k-rule was used for a sum of k distinct shifts. In the article, two particular cases of our main result Theorem 3 were proven. We prove Theorem 3 in a long series of simple lemmas, starting with the fact that every

CA that is a group homomorphism is a sum of symbol endomorphisms. For simplicity, we use additive notation for all groups, as we will see very soon, in Lemma 10, that a full group shift that admits a color blind homomorphic CA must be abelian.

Lemma 7. *Let G be a finite group and let $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be a homomorphic CA with neighborhood $N = [-r, r]$. For all $i \in N$, there exists a group endomorphism $f_i : G \to G$ such that*

- $f_i(g) + f_j(h) = f_j(h) + f_i(g)$ *whenever $h, g \in G$ and $i \neq j \in N$, and*
- $f_{\mathrm{loc}}(g_1, \ldots, g_n) = f_1(g_1) + f_2(g_2) + \cdots + f_n(g_n)$ *for all $g_1, \ldots, g_n \in G$.*

Note that the order of summation in the above formula for f_{loc} is irrelevant by the first item.

Proof. For all $i \in N$, define the function $f_i : G \to G$ by

$$f_i(g) = f_{\mathrm{loc}}(\underbrace{0, \ldots, 0}_{i-1}, g, \underbrace{0, \ldots, 0}_{m-i}),$$

and note that this is an endomorphism of G. Let $i < j \in N$ and $g, h \in G$. Since f_{loc} is a homomorphism, we have

$$\begin{aligned} f_i(g) + f_j(h) &= f_{\mathrm{loc}}(0, \ldots, g, \ldots, 0, \ldots, 0) + f_{\mathrm{loc}}(0, \ldots, 0, \ldots, h, \ldots, 0) \\ &= f_{\mathrm{loc}}(0, \ldots, g, \ldots, h, \ldots, 0) \\ &= f_{\mathrm{loc}}(0, \ldots, 0, \ldots, h, \ldots, 0) + f_{\mathrm{loc}}(0, \ldots, g, \ldots, 0, \ldots, 0) \\ &= f_j(h) + f_i(g), \end{aligned}$$

and for all $g_1, \ldots, g_n \in G$,

$$\begin{aligned} f_{\mathrm{loc}}(g_1, \ldots, g_n) &= \sum_{i=-r}^{r} f_{\mathrm{loc}}(\underbrace{0, \ldots, 0}_{i-1}, g_i, \underbrace{0, \ldots, 0}_{m-i}) \\ &= f_1(g_1) + f_2(g_2) + \cdots + f_n(g_n). \end{aligned}$$

This concludes the proof. □

We call the endomorphisms f_i the *symbol endomorphisms of f*. If $n \geq 1$, all endomorphisms of $\mathbb{Z}_n$ are multiples of the identity map, so we have the following.

Lemma 8. *Suppose that G is a finite abelian group with decomposition $G = \prod_{i=1}^{m} \mathbb{Z}_{p_i^{m_i}}$, where the p_i are prime numbers and $m_i \geq 1$. Then every homomorphic cellular automaton on $G^{\mathbb{Z}}$ is a sum of shifts if and only if the primes p_i are distinct.*

Thus, in general every group homomorphic CA is a *sum of shifted endomorphisms*, and for certain abelian groups the endomorphisms can be taken to be identity maps. Note that the fact that the images of distinct symbol endomorphisms commute means that the local rule of a homomorphic cellular automaton first projects its inputs to subgroups of G which commute with each other, and then multiplies them together. In particular, we have the following.

Lemma 9. *Let G be a group and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. If at least two of the symbol endomorphisms of f are surjective, then G is abelian.*

We now see that in the case of color blind homomorphic CA, there is no loss of generality in retricting to the abelian case.

Lemma 10. *Let G be a finite group and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be color blind and homomorphic with minimal neighborhood size at least 2. Then G is abelian, and if $|G| \geq 4$, then f is a sum of distinct shifts.*

Proof. All groups of order at most 3 are abelian, so we may assume $|G| \geq 4$. Let $0 \neq g \in G$, and consider the configuration $z(g) = {}^{\infty}0g0^{\infty}$. Since the local rule sees at most two distinct symbols in its neighborhood, the image $f(z(g))$ must also be a configuration over $\{0, g\}$ by Lemma 1. Since f commutes with the transposition $(g\ h)$, we have $I = \{i \in \mathbb{Z} \mid f(z(g))_i = g\} = \{i \in \mathbb{Z} \mid f(z(h))_i = h\}$ for all $0 \neq h \in G$. From this we deduce that the symbol endomorphisms of f are either trivial or identity maps, and since at least two of them must be nontrivial, G is abelian by Lemma 9. Also, we clearly have $f = \sum_{i \in N} \sigma^i$, where $N \subset \mathbb{Z}$ is the set of those i for which the symbol endomorphism f_i is nontrivial, so f is a sum of distinct shifts. $\square$

From now on, all alphabets will be abelian groups, so we no longer state this explicitly. Lemma 8 and Lemma 10 now give us the following.

Corollary 3. *Let G be a finite abelian group and $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ a color blind homomorphic CA. Then f is a sum of shifts, which are distinct if $|G| \geq 4$.*

The radius-1 CA f with local rule $(a, b, c) \mapsto a + 2b + c$ is an example of a color blind homomorphic CA on $\mathbb{Z}_3^{\mathbb{Z}}$ which is not a sum of distinct shifts.

Lemma 11. *Let G be a finite abelian group and $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ a homomorphic CA. The f commutes with the symbol permutation $\phi_g(h) = h + g$ if and only if $f({}^\infty g^\infty) = {}^\infty g^\infty$.*

Proof. Having $f({}^\infty g^\infty) = {}^\infty g^\infty$ is equivalent to $f(x) + {}^\infty g^\infty = f(x) + f({}^\infty g^\infty)$ for all $x \in G^{\mathbb{Z}}$, which is simply commutation with ϕ_g, since $f(x) + f({}^\infty g^\infty) = f(x + {}^\infty g^\infty)$. □

We now proceed with a case analysis on the small groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2^2$.

Lemma 12. *Let the CA $f : \mathbb{Z}_2^{\mathbb{Z}} \to \mathbb{Z}_2^{\mathbb{Z}}$ be homomorphic with minimal neighborhood size $m \in \mathbb{N}$. Then f is color blind if and only if f fixes ${}^\infty 1^\infty$, if and only if m is odd.*

Proof. The only nontrivial permutation of $\mathbb{Z}_2$ is ϕ_1, so from Lemma 11 it follows that f is color blind if and only if it fixes ${}^\infty 1^\infty$. Since f is a sum of shifts by Corollary 3, and the shifts are trivially distinct, this is the case if and only if m is odd. □

Lemma 13. *Denote $G = \mathbb{Z}_2^2$, and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. Then f is color blind iff it is a sum of an odd number of distinct shifts.*

Proof. The proof relies on the facts that $2ng = 0$ for all $g \in G$ and $n \in \mathbb{N}$, and if $G = \{a, b, c, d\}$ then $a + b + c = d$.

Suppose first that f is color blind. Corollary 3 applies, so that f is a sum of m distinct shifts for some $m \in \mathbb{N}$. This means that $X = \{(0, 0), (0, 1)\}^{\mathbb{Z}} \cong \mathbb{Z}_2^{\mathbb{Z}}$ is closed under f, and the restriction of f to X is also a sum of shifts. If $f|_X$ were not color blind then f would not be either, so m must be odd by Lemma 12.

On the other hand, let f be a sum of m distinct shifts for odd m, and consider an arbitrary transposition $\phi = (g\ h)$. Denote $G = \{a, b, g, h\}$. Let $g_1, \ldots, g_m \in G$, and for $c \in G$, let n_c be the number of $i \in \{1, \ldots, m\}$ such that $g_i = c$.

If both n_g and n_h are even, then exactly one of n_a and n_b is odd, let us say n_a. Then $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$. If both n_g and n_h are odd, we may again assume n_a is odd and n_b is even, so that $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a + g + h = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$, since $a + g + h = b \notin \{g, h\}$ is a fixed point of ϕ.

If $n_g + n_h$ is odd, we may assume n_g is odd and n_h is even. Then, $n_a + n_b$ is even, and the cases left to consider are that both n_a and n_b are odd or both are even. If n_a and n_b are both odd, then $f_{\mathrm{loc}}(g_1, \ldots, g_m) = a + b + g = h$, which implies $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a + b + h = g$ and $\phi(f_{\mathrm{loc}}(g_1, \ldots, g_m)) = \phi(h) = g$. If both are even, then $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = h = \phi(g) = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$. This finishes the proof since transpositions generate the group of permutations. $\square$

Lemma 14. *Let the CA $f : \mathbb{Z}_3^{\mathbb{Z}} \to \mathbb{Z}_3^{\mathbb{Z}}$ be homomorphic. Then f is color blind if and only if it fixes ${}^\infty 1^\infty$, if and only if it is a sum of $3k+1$ shifts for some k.*

Proof. By Lemma 11, f fixes ${}^\infty 1^\infty$ if and only if it commutes with the symbol permutation ϕ_1. We prove that all such homomorphic CA are color blind, for which it is enough to show that they also commute with the transposition $(1\ 2)$. By Corollary 3, f is a sum of shifts $\sum_{i=1}^m \sigma^{k_i}$ for some $m \in \mathbb{N}$ and $k_i \in \mathbb{Z}$. For all $x \in \mathbb{Z}_3^{\mathbb{Z}}$, we then have

$$((1\ 2) \circ f)(x) = (1\ 2)\left(\sum_{i=1}^m \sigma^{k_i}(x)\right) = \sum_{i=0}^m (1\ 2)(\sigma^{k_i}(x)))$$
$$= \sum_{i=0}^m \sigma^{k_i}((1\ 2)(x)) = (f \circ (1\ 2))(x),$$

where the second equality follows from the fact that $(1\ 2)$ is an automorphism of $\mathbb{Z}_3$ and the third one directly from the fact that $(1\ 2)$ is a cellular automaton.

Finally, it is easy to see that a sum of m shifts on $\mathbb{Z}_3^{\mathbb{Z}}$ fixes the point ${}^\infty 1^\infty$ if and only if $m \equiv 1 \bmod 3$. $\square$

Finally, we handle the remaining cases in a single lemma.

Lemma 15. *Let G be a finite abelian group such that $|G| > 3$ and $G \ncong \mathbb{Z}_2^2$, and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. Then f is color blind if and only if it is a shift map.*

Proof. First, a shift map is trivially a color blind homomorphic CA for any group alphabet.

As for the nontrivial direction, Corollary 3 again applies, so that f_{loc} returns the sum of the values in the neighborhood N of f. If $|N| = 0$, then f does not commute with symbol permutations, as it sends everything to ${}^\infty 0^\infty$. Assume then that $|N| \geq 2$.

We first suppose $|G| > 4$. In this case, we take $0 \neq g \in G$ and $h \in G$ such that $h \notin \{0, g, -g\}$. Now, $g + h \notin \{0, g, h\}$, so that $f_{\mathrm{loc}}(g, h, 0, \ldots, 0) = g + h \notin \{0, g, h\}$, which is a contradiction by Lemma 1. Now, let $|G| =$

4, so by the assumption that $G \not\cong \mathbb{Z}_2^2$, we have that $G \cong \mathbb{Z}_4$. But now $f_{\rm loc}(1, 1, 0, \ldots, 0) = 2$, again contradicting Lemma 1.

Of course, in the remaining case that $|N| = 1$, f is a shift map. $\square$

We collect the results of this section into a single statement.

Theorem 3. *Let G be a finite group, and let $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be a homomorphic cellular automaton. Then, f is color blind iff one of the following (partially overlapping) conditions holds.*

- *$G = \mathbb{Z}_2$, $G = \mathbb{Z}_2^2$ or $G = \mathbb{Z}_3$, and f fixes unary points,*
- *$G = \mathbb{Z}_2$ or $G = \mathbb{Z}_2^2$, and f is a sum of an odd number of distinct shifts,*
- *$G = \mathbb{Z}_3$, and f is a sum of $3k + 1$ shifts for some k,*
- *$|G| > 4$ or $G = \mathbb{Z}_4$, and f is a shift map.*

Proof. If G is not abelian, then f is a shift map by Lemma 10. In the converse case, Lemma 12, Lemma 13, Lemma 14 and Lemma 15 give the claim. $\square$

This gives a complete characterization of homomorphic color blind cellular automata on full shifts whose alphabet is a finite group. We also note that in our arguments we mainly manipulated the local functions of cellular automata, so the result should hold as such for multidimensional automata with exactly the same proofs. Thus Theorem 3 is a generalization of the results of [10], which state that for all dimensions $d \geq 1$, any sum of 4 distinct shifts on $\mathbb{Z}_3^{\mathbb{Z}^d}$ is color blind, and no sum of m distinct shifts on $\mathbb{Z}_n^{\mathbb{Z}^d}$ is color blind if $n \geq m > 1$.

7 EQUICONTINUOUS AND ALMOST EQUICONTINUOUS FAMILIES

In this section, we show a connection between the centralizer of an equicontinuous family of cellular automata and Π-blind cellular automata (including the centralizer of a single equicontinuous cellular automaton), and also discuss almost equicontinuous families. The results of this section are a little more abstract then the previous ones, even though their proofs are still largely combinatorial, and we state them in the usual topological framework.

Definition 7. *Let $F = \{f_1, \ldots, f_k\}$ be a finite family of cellular automata on a subshift X. A point $x \in X$ is called an* equicontinuity point for F *if*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall f \in F^*, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon. \tag{2}$$

If every point $x \in X$ is an equicontinuity point for F, then we say F is equicontinuous. *If the equicontinuity points are dense in X, then we say F* is almost equicontinuous.* *We say F is reversible if every cellular automaton f_i is.*

An obvious example of an equicontinuous family of maps on $S^{\mathbb{Z}}$ is the set of symbol permutations (symbol maps), and the centralizer of that set is exactly the family of color blind (typhlotic) cellular automata on $S^{\mathbb{Z}}$. If X and Y are subshifts, $\alpha : X \to Y$ is a conjugacy between them and $f : X \to X$ is a cellular automaton on X, then we write $\alpha f : Y \to Y$ for the map $\alpha f = \alpha \circ f \circ \alpha^{-1}$.

Proposition 4. *If F is an equicontinuous family of maps on a subshift X, then there exists a subshift Y and a conjugacy $\alpha : X \to Y$ such that αf is a symbol map for all $f \in F^*$.*

This is a special case of Theorem 2 of [13], but we include a proof for completeness.

Proof. Let $\epsilon = 1$, and let $r \in \mathbb{N}$ be such that $\delta > 2^{-r}$ for the number $\delta > 0$ given by (2) for the family F and ϵ. Then every automaton in F^* has radius r, by the definition of the metric d.

Define the equivalence relation $\sim$ on $\mathcal{B}_{2r+1}(X)$ by

$$v \sim w \iff \forall f \in F^* : f_{\mathrm{loc}}(v) = f_{\mathrm{loc}}(w)$$

for all $v, w \in \mathcal{B}_{2r+1}(X)$, and denote $B = \mathcal{B}_{2r+1}(X)/\sim$. Define the block map $\alpha : X \to B^{\mathbb{Z}}$ by the radius-r local rule that sends each block to its $\sim$-equivalence class. Note that since F^* contains the identity automaton of radius r, $v \sim w$ implies $v_r = w_r$ for all $v, w \in \mathcal{B}_{2r+1}(X)$, and thus α is injective. We have constructed a conjugacy between X and $Y = \alpha(X)$.

It remains to be shown that $\alpha f = \alpha \circ f \circ \alpha^{-1}$ is a symbol map whenever $f \in F^*$, and for that, let $x, y \in X$ be such that $\alpha(x)_0 = \alpha(y)_0$. This means that $x_{[-r,r]} \sim y_{[-r,r]}$, and thus $f(x)_0 = f(y)_0$. If we now had $f(x)_{[-r,r]} \not\sim$

* So F is (almost) equicontinuous if the action of the free monoid on $|F|$ generators given by the maps F is (almost) equicontinuous.

$f(y)_{[-r,r]}$, there would exist some $g \in F^*$ with $g(f(x))_0 \neq g(f(y))_0$. But this is a contradiction, since $x_{[-r,r]} \sim y_{[-r,r]}$ and $g \circ f \in F^*$, so we have $f(x)_{[-r,r]} \sim f(y)_{[-r,r]}$, and thus $\alpha(f(x))_0 = \alpha(f(y))_0$, which finishes the proof. □

We now show an application of the theory developed so far, in the sense of giving some kind of structure for the centralizer of an equicontinuous family of maps. Let F be a reversible equicontinuous family of cellular automata on a sofic shift $X \subset S^{\mathbb{Z}}$. By Proposition 4, we may recode the family F into a set Π of symbol permutations on another (automatically sofic) subshift $Y \subset T^{\mathbb{Z}}$. Now, understanding the centralizer of F on X amounts to understanding the Π-blind maps on Y. These, in turn, are exactly the restrictions to Y of those Π-blind cellular automata on $T^{\mathbb{Z}}$ that keep Y invariant, by Proposition 1.

As a concrete implementation of this idea, we prove the following generalization of Theorem 1.

Proposition 5. *Let F be a reversible equicontinuous family of cellular automata on $S^{\mathbb{Z}}$. Then the centralizer of F contains an intrinsically universal cellular automaton.*

Proof. First, we apply Proposition 4 to F, and obtain a conjugacy α from $S^{\mathbb{Z}}$ to a nontrivial mixing SFT $X \subset T^{\mathbb{Z}}$ that sends F to a subset Π of the symbol permutations on $T^{\mathbb{Z}}$. Note that X is closed under Π^*, and the centralizer of F corresponds to the set of Π-blind maps on X. Let $m \in \mathbb{N}$ be the window size and mixing distance of X.

Next, we show by a standard technique that we can forbid some finite set of words from X, including a set closed under Π^*, and obtain another nontrivial mixing SFT $Y \subsetneq X$. Namely, it is known (see for example Section 9.4 of [8]) that there exists a shift-invariant probability measure μ on X and $c, \lambda > 0$ such that for any $w \in \mathcal{B}(X)$, we have $\mu(\{x \in X \mid x_{[0,|w|-1]} = w\}) < c\lambda^{|w|}$. Since Π^* is finite, there is a set $W \subset \mathcal{B}_{3k}(X)$ closed under Π^*, with $k \geq m+1$, such that $\mu(\bigcup_{i=-k}^{3k-1} C_i) < 1$, where

$$C_i = \{x \in X \mid \exists w \in W : x_{[i,i+k-1]} \in \{w_{[0,k-1]}, w_{[2k,3k-1]}\}\}.$$

Let $x \in X \setminus \bigcup_{i=-k}^{3k-1} C_i$, and denote $u = x_{[0,3k-1]}$. We may assume that u contains every symbol of T. Define $Y \subsetneq X$ as the set of configurations $y \in X$ where the distance between two consecutive occurrences of u (indices $i \in \mathbb{Z}$ with $y_{[i,i+3k-1]} = u$) is at most $3k+m+1$. Then no $w \in W$ occurs in any $y \in Y$, for otherwise one of the u occurring in y would contain $w_{[0,k-1]}$ or $w_{[2k,3k-1]}$, contradicting the choice of u. Now Y is a mixing SFT, since the distances between consecutive occurrences of u can be independently chosen from at least m and $m+1$, and it is nontrivial since u contains every symbol of T.

Let $M \geq \max(4k, r)$ be a window size and mixing distance for Y, where $r \in \mathbb{N}$ is a radius for α and α^{-1}, and let $v \in \mathcal{B}_M(Y)$ and $w \in W$. There exist words $v_\ell, v_r \in \mathcal{B}_m(X)$ such that $v' = vv_\ell wv_r v \in \mathcal{B}(X)$. Deconstruct the word as $v' = tw't'$, where $w' \in W$ and $|t| \geq k$ is minimal. There also exist distinct $a, b \in \mathcal{B}(Y)$ of the same length (at least $2M$) such that $vav, vbv \in \mathcal{B}(Y)$, and then any configuration formed by concatenating the words $v'a$ and $v'b$ is in X. Denote by $Z \subset X$ the subshift of these configurations. For any $z \in Z$ and $i \in \mathbb{Z}$, we now have $z_{[i,i+|v'|-1]} = v'$ if and only if $z_{[i+j,i+j+3k-1]} \in W$ holds for $j = |t|$ but not for any $j \in [|t| - 2M, |t| - 1]$. The same property holds also for the occurrences of $\pi(v')$ in $\pi(z)$, for any $\pi \in \Pi^*$, since W is closed under the action of Π^*. Since every symbol of T occurs in v', no two elements of Π^* map it to the same word, and thus the subshifts $\pi(Z)$ for $\pi \in \Pi^*$ are pairwise disjoint.

Let $f : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be an intrinsically universal CA, and define $g : Z \to Z$ to be the automaton that applies f to the concatenations of $v'a$ and $v'b$, treating them as 0 and 1, respectively. Since the subshift $\pi(Z)$ is disjoint from Z for any nontrivial $\pi \in \Pi^*$, the automaton g is trivially Π-blind on Z. We then apply Proposition 2 to extend g into a Π-blind automaton $\hat{g} : X \to X$. The conditions of the proposition are satisfied, since every symbol of T occurs in every configuration of Z. Then $h = \alpha^{-1}\hat{g}$ is a cellular automaton on $S^{\mathbb{Z}}$ that commutes with every element of F. Since we chose $|a| = |b| \geq 2r$, every configuration in $\alpha^{-1}(Z)$ is an infinite concatenation of some words a' and b' of the same length (that of $v'a$ and $v'b$), and that h behaves as f on those configurations, treating the words as 0 and 1, respectively. This implies that h is intrinsically universal. □

For almost equicontinuous families, things become more complicated, and we have not been able to obtain the result in the same generality. However, if we add some additional constraints, things become tangible again. The following definition has also appeared in [14].

Definition 8. *Let F be a finite set of cellular automata on a subshift X. A nonempty set of words $W \subset \mathcal{B}_\ell(X)$ is* visibly blocking *if*

- *for all configurations $x \in X$ and $f \in F^*$ we have $x_{[0,\ell-1]} \in W$ iff $f(x)_{[0,\ell-1]} \in W$, and*
- *for all configurations $x, y \in X$ such that $x_{[0,\ell-1]} \in W$ and $x_{[0,\infty)} =$*

$y_{[0,\infty)}$ ($x_{(-\infty,\ell-1]} = y_{(-\infty,\ell-1]}$), we have $f(x)_{[\ell,\infty)} = f(y)_{[\ell,\infty)}$ ($f(x)_{(-\infty,-1]} = f(y)_{(-\infty,-1]}$, respectively) for all $f \in F^*$.

We denote by $\chi_W : X \to \{0,1\}^{\mathbb{Z}}$ the map defined by $\chi_W(x)_i = 1 \iff x_{[i,i+\ell-1]} \in W$. By the above, we have $\chi_W = \chi_W \circ f$ for all $f \in F^*$.

Every finite family of CA on a transitive SFT that has a visibly blocking set W is almost equicontinuous, as any configuration containing infinitely many elements of W to the left and right is an equicontinuity point for the family.

Example 6. *Let $S = \{0,1,2\}$, and define the cellular automaton f on $S^{\mathbb{Z}}$ by the neighborhood-$[0,1]$ local function $f_{\mathrm{loc}}(a,b) = a + b \bmod 2$ and $f_{\mathrm{loc}}(2,b) = 2$ for all $a \in \{0,1\}$ and $b \in S$. Then $\{2\}$ is a visibly blocking set for the singleton family $\{f\}$. Also, any infinite concatenation of the words 02 and 12 is a fixed point of f, and using them, we can construct an intrinsically universal CA in the centralizer of f. Namely, we may simulate any binary CA on these configurations, and act as the identity on all other local patterns.*

Not all almost equicontinuous CA possess visibly blocking sets, and a classical counterexample is provided by the *Coven CA* (intruduced in [4], see [1] for formulation and proof of the property). For those that do, we have the following generalization of both Proposition 5 and the above example.

Theorem 4. *Let F be a finite family of finite-to-one cellular automata on $S^{\mathbb{Z}}$, and let $W \subset S^\ell$ be a visibly blocking set for F. Then the centralizer of F contains an intrinsically universal automaton.*

Proof. First, if every configuration in $S^{\mathbb{Z}}$ contains occurrences of words in W, then they always occur with bounded gaps, and thus F is equicontinuous. Then we can apply Proposition 5 to obtain the universal automaton. Thus we assume that the SFT $X \subset S^{\mathbb{Z}}$ where no words of W occur is nonempty, and by the properties of W, we have $f(X) \subset X$.

If X is finite, then it consists only of periodic configurations, and since $\chi_W = \chi_W \circ f$ for all $f \in F^*$, long periodic patterns of X cannot be erased by the automata in F. This again implies that F is equicontinuous, so we may assume that X is infinite. Then there exist $a \neq b \in \mathcal{B}_{4\ell}(X)$ that have the same prefixes and suffixes of length ℓ (we may increase ℓ is necessary). Choose any $w \in W$, and denote $u = waw$ and $v = wbw$. Denote by $Y \subset S^{\mathbb{Z}}$ the subshift of infinite concatenations of u and v, and note that Y has a global period of 6ℓ induced by the positions of the occurrences of u and v. In fact, there exists a periodic configuration $\eta \in \{0,1\}^{\mathbb{Z}}$ with least period 6ℓ such that

$$\chi_W(Y) = \{\sigma^i(\eta) \mid i = 0, \ldots, 6\ell - 1\}$$

For $f \in F^*$, denote $f(u) = f({}^\infty u.u^\infty)_{[0,6\ell-1]}$. Since W is visibly blocking, we actually have $f(x)_{[0,6\ell-1]} = f(u)$ for any $x \in Y$ with $x_{[0,6\ell-1]} = u$. The analogous claim holds for v. Also, since every $f \in F^*$ is finite-to-one, it is also reversible on the subshift $Z = \bigcup_{f \in F^*} f(Y)$.

Let $g : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be an intrinsically universal color blind cellular automaton given by Theorem 1. Because of the global period of Y, g can be simulated by an automaton $\tilde{g} : Y \to Y$ on the words u and v, and it satisfies $\tilde{g}(f(y)) = f(\tilde{g}(y))$ for every $y \in Y$ such that $f(y) \in Y$. By the same argument as in the proof of Proposition 1, we can extend $\tilde{g}$ to Z so that it commutes with the restriction of every $f \in F^*$ to Z. Let $r = m\ell \geq 6\ell$ be a radius for $\tilde{g}$, and define an auxiliary shift-commuting map $\xi : S^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ as follows. For $x \in S^{\mathbb{Z}}$, define $\xi(x)_0 = 1$ if and only if $x_{[-r-\ell,6\ell+r+\ell-1]} = w'cdew''$, where $d = f(u)$ or $d = f(v)$ for some $f \in F^*$, and $w', w'' \in W$ are such that $f'(x)_{[-r,6\ell+r-1]} \in \mathcal{B}(Z)$ for all $f' \in F^*$ (since W is visibly blocking, this only depends on the word $w'cdew''$). Note that $\xi = \xi \circ f$ for any $f \in F^*$. Define now the CA $h : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ on any $x \in S^{\mathbb{Z}}$ by

$$h(x)_0 = \begin{cases} \tilde{g}_{\mathrm{loc}}(x_{[-r,r]}), & \text{if } \xi(x)_i = 1 \text{ for some } i \in [-6\ell+1, 0], \\ x_0, & \text{otherwise.} \end{cases}$$

We say that 0 is an *amenable coordinate for* x, if the first condition holds.

The automaton h is now intrinsically universal, since $h_Y = \tilde{g}$, so we only need to show that it commutes with every $f \in F$. Let thus $x \in S^{\mathbb{Z}}$ be arbitrary. If 0 is not amenable for x, then it is not amenable for $f(x)$ either, and we have $h(f(x))_0 = f(x)_0 = f(h(x))_0$, since h behaves nontrivially only on amenable coordinates that occur between W-words, and these changes cannot propagate under the action of F. On the other hand, suppose that 0 is amenable for x, and thus also for $f(x)$. Then the central coordinates of x are of the form $w'cdew''$ as above and since W is visibly blocking, we may in fact assume that $x \in Z$, and thus $h(x) = \tilde{g}(x)$. But then it follows immediately that $h(f(x))_0 = f(h(x))_0$. □

For a general finite-to-one almost equicontinuous family, this construction cannot be carried out as such, and we leave this problem open. Note that the symbol maps form an equicontinuous, and thus almost equicontinuous, family of cellular automata whose commutator consists exactly of the shift maps if the alphabet size exceeds 2. Thus some further restriction, like being finite-to-one or reversible, is necessary.

Question 1. *Does the centralizer of every finite almost equicontinuous family of finite-to-one (or reversible) cellular automata contain an intrinsically* *universal one?*

In the article [11], it was shown that the centralizer of a cellular automaton which is *bipermutive* (permuting the left- or rightmost coordinate of the neighborhood also permutes the image of the local rule) and *affine* (group homomorphic up to a constant) contains only affine CA, which cannot be intrinsically universal. Furthermore, in [15] it was shown that the centralizer of a bipermutive CA has zero density, but the existence of an intrinsically universal automaton was left open. This problem can be seen as opposite to the above results, since bipermutive cellular automata are, in a sense, as far away from being equicontinuous as possible.

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