Characterizing Asymptotic Randomization in Abelian Cellular Automata

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Abstract. Abelian cellular automata (CA) are CA which are group endomorphisms of the full group shift when endowing the alphabet with an abelian group structure. A CA randomizes an initial probability measure if its iterated images weak*-converge towards the uniform Bernoulli measure (the Haar measure in this setting). We are interested in structural phenomena, i.e. randomization for a wide class of initial measures (under some mixing hypotheses). First, we prove that an abelian CA randomizes in Cesàro mean if and only if it has no soliton, i.e. a finite configuration whose time evolution remains bounded in space. This characterization generalizes previously known sufficient conditions for abelian CA with scalar or commuting coefficients. Second, we exhibit examples of strong randomizers, i.e. abelian CA randomizing in simple convergence; this is the first proof of this behaviour to our knowledge. We show however that no CA with commuting coefficients can be strongly randomizing. Finally, we show that some abelian CA achieve partial randomization without being randomizing: the distribution of short finite words tends to the uniform distribution up to some threshold, but this convergence fails for larger words. Again this phenomenon cannot happen for abelian CA with commuting coefficients.

1. *Introduction*

Cellular automata, although extremely simple to define, provide a rich source of examples of dynamical systems which are not well understood yet. This is in particular true when taking a measure theoretic point of view and studying the evolution of a probability measure under iterations of a CA. The situation can be roughly depicted as follows: for non-surjective CAs essentially all behaviors that are not prohibited by immediate computability restrictions can happen [**5, 3, 7**]; for the surjective case various forms of rigidity are observed (see [**20**] for an overview). In particular, since the pioneering work of Lind and Miyamoto on the 'addition modulo 2' CA [**19, 16**], many CAs of algebraic origin where shown to behave like randomizers [**9, 21, 22, 18, 13**], *i.e.* they converge in Cesàro mean or in density to the uniform Bernoulli measure from any initial probability measure from a large class $\mathcal{C}$. In [**19, 16**], the class $\mathcal{C}$ is that of Bernoulli measures of full support. It was later extended to full support Markov measures or N-step Markov processes, measures with complete connections and summable decay of correlations and harmonically mixing measures [**20**] and more [**23, 26**].

Apart from specific examples (like in [**17**]), the class of CA where randomizing behavior has been shown is essentially contained in that of 'linear' CA defined on an abelian group alphabet by

$$F(x)_i = \sum_{j\in V} \theta_j(x_{i+j})$$

where θ_j are *commuting* endomorphisms (most of the time automorphisms or scalar coefficients). Furthermore, the type of convergence considered has always been Cesàro mean or convergence in density.

In this paper we consider the class of harmonically mixing measures and the class of abelian CAs which are like 'linear' CA described above but *without* the assumption of commutation of endomorphisms. Our first main result is a complete characterization of randomization in density in that setting.

> THEOREM — *An abelian CA F randomizes in density any harmonically mixing measure if and only if it does not possess a soliton, i.e. a finite configuration whose set of non-zero cells stays with a bounded diameter under iterations of the CA.*

We show that this theorem extends the most general previous result [**22**] and allows to easily prove randomization on particular examples, even in the setting of non-commutative coefficients [**17**], which answers a question of [**20**].

Our approach uses tools from harmonic analysis with a similar approach to the work of Pivato and Yassawi on diffusion of characters [**21, 22**]. We rely on the abelian structure of the considered CA to reduce randomization to a combinatorial property of diffusivity. More precisely, we define a dual CA on the (Pontryagin) dual group, show that diffusion in the dual is equivalent to randomization and that the diffusion property is preserved by duality. Finally, we prove the equivalence between diffusivity and the absence of solitons, not using the abelian structure but general combinatorial properties of surjective CA. This allows us to go beyond the commuting coefficient case, which was treated in previous works by a careful analysis of binomial coefficients of the iterates of F.

We also prove the existence of a stronger form of randomization where taking subsequences of density 1 or Cesàro mean is not necessary:

Theorem — *There exist abelian CAs that randomize in simple convergence any harmonically mixing measure.*

This answers a question of **[14]** (Question 59). Experiments on small surjective CAs **[12, 27]** suggest that this strong form of randomization is the most common, that it occurs as well on non-abelian CA, and even that randomization occurring only in density (or Cesàro mean) might be an artifact of abelian CAs. This confirms the importance of the non-commutative coefficients case, since we also prove that abelian CA with commuting coefficients cannot achieve such a strong form of randomization.

The results above are stated as randomization for the class of harmonically mixing measures. We do not investigate when there are more randomized measures (like in **[23]**), however we show that there can not be less: if an abelian CA randomizes in density full-support Bernoulli measures, then it randomizes in density all harmonically mixing measures. Interestingly, the rigidity is even stronger for abelian CA with commutative coefficients: we prove that if the frequency of individual states is randomized (in density), then the CA is fully randomizing in density. In the case of non-commutative coefficients we can have partial randomization: we give examples for any K of abelian CA which do randomize all cylinders up to size K, but fail to randomize completely. This suggests that experimental work on randomization in general CA should be done with care: randomization might fail in non-obvious ways on long-range correlations.

The paper is organized as follows: in Section 2 we recall basic definitions and tools about measure theoretic aspects of cellular automata; in Section 3 we study the evolution of ranks of characters under iterations of abelian CAs, the property of character diffusion and its link with randomization; in Section 4 we define the dual of an abelian CA, show that duality preserves diffusivity in density and link this property with the absence of solitons; in Section 5 we establish our main result, a characterization of randomization in density through the pre sence of solitons; in Section 6, we exhibit a class examples of strong randomization and randomization up to fixed-length cylinder, and we show that these behaviors are specific to CA with non-commuting coefficients; finally, in Section 7 we give some directions for further research on this topic.

2. *Definitions and tools*

In all the paper we give our results for dimension 1 but all results extend straightforwardly to the d-dimensional case.

Let $\mathcal{A}$ be a finite alphabet. We define $\mathcal{A}^* = \bigcup_n \mathcal{A}^n$ the set of finite *words*, and $\mathcal{A}^{\mathbb{Z}}$ the set of (one-dimensional) *configurations*. For a finite set $U \subset \mathbb{Z}$ and $u \in \mathcal{A}^U$, define the *cylinder*:

$$[u]_U = \{x \in \mathcal{A}^{\mathbb{Z}} \ : \ x|_U = u\}.$$

For $u \in \mathcal{A}^n$ and $k \in \mathbb{Z}$, also define $[u]_k = [u]_{\{k,\dots,k+n-1\}}$ and $[u] = [u]_0$.

We endow $\mathcal{A}^{\mathbb{Z}}$ with the product topology, which is metrizable using the *Cantor distance*:

$$\forall x, y \in \mathcal{A}^{\mathbb{Z}}, d(x,y) = 2^{-\Delta(x,y)} \quad \text{where} \quad \Delta(x,y) = \min\{|i| : x_i \neq y_i\}.$$

The *shift map* is defined by

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, \sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}.$$

A *cellular automaton* is a pair $(\mathcal{A}, F)$ where $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a continuous function that commutes with the shift (i.e. $F \circ \sigma = \sigma \circ F$). Equivalently, F is defined by a finite neighborhood $\mathcal{N} \subset \mathbb{Z}$ and a *local rule* $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$ in the sense that:

$$\forall x \in \mathcal{A}^{\mathbb{Z}}, \forall i \in \mathbb{Z}, F(x)_i = f(x_{i+\mathcal{N}}).$$

If 0 denotes the neutral element of the group $(\mathcal{A}, +)$, a *finite configuration* is a configuration $x \in \mathcal{A}^{\mathbb{Z}}$ such that $x(i) = 0$ for all $i \in \mathbb{Z}$ except a finite set. If x is a finite configuration, we define its *support* by $\mathrm{supp}(x) = \{i \in \mathbb{Z} : x(i) \neq 0\}$ and its *rank* by $\mathrm{rank}(x) = |\,\mathrm{supp}(x)|$. Note that the set of finite configurations is dense in $\mathcal{A}^{\mathbb{Z}}$.

An *abelian cellular automaton* F is a cellular automaton which is an endomorphism for $(\mathcal{A}^{\mathbb{Z}}, \overline{+})$:

$$\forall x, y \in \mathcal{A}^{\mathbb{Z}}, F(x \overline{+} y) = F(x) \overline{+} F(y)$$

Equivalently, F is a finite sum of shifts composed with endomorphisms of $(A, +)$. More precisely, there is a finite $\mathcal{N} \subset \mathbb{Z}$ and a collection $(\phi_i)_{i\in\mathcal{N}}$ of endomorphisms of $(A, +)$ such that:

$$F = \sum_{i\in\mathcal{N}} \overline{\phi}_i \circ \sigma^i, \qquad \text{where} \qquad \overline{\phi}_i : \begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} & \to & \mathcal{A} \\ x & \mapsto & \big(\phi_i(x(j))\big)_j \end{array}.$$

Note that the image of a finite configuration is always a finite configuration.

We say that F has *commuting endomorphisms* if the endomorphisms ϕ_i commute pairwise. Supposing $\mathcal{A}$ is a vector space over a finite field $\mathbb{F}_p$ turns out to be a source of simple yet illustrative examples. As said above, we are particularly interested in the non-commuting case and we will illustrate our results with the two representatives F_2 and H_2 defined over $\mathcal{A} = \mathbb{F}_2^2$ by

$$F_2(x)_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i+1}$$

$$H_2(x)_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i-1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_i + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot x_{i+1}$$

where elements of $\mathcal{A}$ are seen as vectors and matrix notation is used to denote endomorphisms of $\mathcal{A}$.

2.1. *Cellular automata acting on probability measures* Let $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ be the space of probability measures on the Borel sigma-algebra of $\mathcal{A}^{\mathbb{Z}}$. In particular, we consider $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ the subset of all σ-invariant measures. Here are a few examples that we mention throughout the paper:

Bernoulli measure Take $(\beta_i)_{i\in\mathcal{A}} \in [0,1]^{\mathcal{A}}$ such that $\sum_i \beta_i = 1$. Let β be the usual Bernoulli measure of parameters (β_i) on $\mathcal{A}$. The *Bernoulli measure* of parameters (β_i) on $\mathcal{A}^{\mathbb{Z}}$ is defined as $\mu = \otimes_{\mathbb{Z}}\beta$; that is, each cell is drawn in a i.i.d manner and distributed as a Bernoulli measure. In other words,

$$\forall u \in \mathcal{A}^*, \mu([u]) = \prod_{0\le i<|u|} \beta_{u_i}.$$

A particularly important example is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$, denoted λ, which is the Bernoulli measure of parameters $\left(\frac{1}{|\mathcal{A}|}\right)_{i\in\mathcal{A}}$.

Markov measure Let $(p_{i,j})_{i,j\in\mathcal{A}^2}$ be a matrix satisfying $\sum_j p_{ij} = 1$ for all i, and $(\mu_i)_{i\in\mathcal{A}}$ an eigenvector associated with the eigenvalue 1 (the choice being unique if the matrix is irreducible). The associated *two-step Markov measure* is defined as

$$\forall u \in \mathcal{A}^*, \mu([u]) = \mu_{u_0} \prod_{0\le i<|u|} p_{u_i u_{i+1}}.$$

This can be extended to n-step Markov measures.

The weak-* topology on $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is metrisable. A possible metric is given by the distance:

$$d_{\mathcal{M}}(\mu,\nu) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} \max_{u\in\mathcal{A}^{2k+1}} |\mu([u]_{-k}) - \nu([u]_{-k})|.$$

A cellular automaton $(\mathcal{A}, F)$ yields a continuous action on the space of probability measures $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$:

$$\text{For any borelian } U,\ F\mu(U) = \mu(F^{-1}U)$$

Considering the iterated action of a cellular automaton on an initial measure μ, we distinguish various forms of convergence:

- $(F^t\mu)_{t\in\mathbb{N}}$ *converges* to ν if $F^t\mu \to \nu$ (for the weak-$*$ convergence); equivalently, $F^t\mu([u]) \to \nu([u])$ for every finite word u;
- $(F^t\mu)_{t\in\mathbb{N}}$ *converges in Cesàro mean* to ν if $\frac{1}{T}\sum_{t=0}^{T-1} F^t\mu \to \nu$; equivalently, if $\frac{1}{T}\sum_{t=0}^{T-1} F^t\mu([u]) \to \nu([u])$ for every finite word u;

- $(F^t\mu)_{t\in\mathbb{N}}$ *converges in density* to ν if there exists an increasing sequence $(\varphi(t))_{t\in\mathbb{N}}$ of lower density 1 such that $F^{\varphi(t)}\mu \to \nu$; equivalently†, if for every finite word u there exists an increasing sequence $(\varphi_u(t))_{t\in\mathbb{N}}$ of lower density 1 such that $F^{\varphi_u(t)}\mu([u]) \to \nu([u])$;

- $(F^t\mu)_{t\in\mathbb{N}}$ *converges on cylinders of support* $\subset \mathbb{U}$ to ν if $\mu(\cdot \mid \mathfrak{B}_\mathbb{U}) \to \nu(\cdot \mid \mathfrak{B}_\mathbb{U})$ where $\mathfrak{B}_\mathbb{U}$ is the Borel σ-algebra generated by the cylinders of support $\subset \mathbb{U}$ (this can be seen as convergence of measures of $\mathcal{M}(\mathcal{A}^\mathbb{U})$); equivalently, $F^t\mu([u]) \to \nu([u])$ for every word u such that $\text{supp}(u) \subset \mathbb{U}$.

Recall that, in a context where the alphabet is $\mathcal{A}$, λ is the uniform Bernoulli measure on $\mathcal{A}^\mathbb{Z}$.

Definition 1 (Randomization) *Let $F : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ be a cellular automaton and $\mathcal{M} \subset \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be a class of initial measures.*

F strongly randomizes $\mathcal{M}$ *(resp. in Cesàro mean, in density, on cylinders of support $\mathbb{U}$) if, for all $\mu \in \mathcal{M}$, $(F^t\mu)_{t\in\mathbb{N}}$ converges to λ (resp. in Cesàro mean, in density, on cylinders of support $\mathbb{U}$).*

Proposition 1. *Strong randomization implies all other forms of randomization, and randomization in Cesàro mean is equivalent to randomization in density.*

Proof. The first point is clear. The second point stems from the fact that the uniform Bernoulli measure λ is an extremal point of $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ and $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is compact. We prove this point by contraposition. Assume that $(F^t\mu)_{t\in\mathbb{N}}$ does not converge to λ in density, i.e. there exists some $\varepsilon > 0$ and some sequence $(\varphi(t))$ of upper density $\alpha > 0$ such that $F^{\varphi(t)}\mu \notin B(\lambda,\varepsilon)$, where $B(\lambda,\varepsilon)$ is the open ball of radius ε centered on λ. Therefore there exists a sequence of times $(T_i)_{i\in\mathbb{N}}$ such that $\frac{T_i}{\varphi(T_i)} \to \alpha$. Then:

$$\frac{1}{\varphi(T_i)+1}\sum_{t=0}^{\varphi(T_i)} F^t\mu = \frac{1}{\varphi(T_i)+1}\sum_{t=0}^{T_i} F^{\varphi(t)}\mu + \frac{1}{\varphi(T_i)+1}\sum_{\substack{t=0\\ t\notin\varphi(\mathbb{N})}}^{\varphi(T_i)} F^t\mu.$$

Let $\mathcal{C}$ be the convex hull of $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})\backslash B(\lambda,\varepsilon)$. By compactness, this sequence admits accumulation points which must be of the form $\alpha\nu + (1-\alpha)\eta$ for some $\nu \in \mathcal{C}$ and $\eta \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$. However, since λ is extremal, $\lambda \notin \mathcal{C}$, so that $\lambda \neq \nu$, and $\lambda \neq \alpha\nu + (1-\alpha)\eta$. In other words, the sequence $(\frac{1}{T+1}\sum F^t\mu)_{t\in\mathbb{N}}$ admits some accumulation point which is not λ, and we conclude. □

2.2. *Fourier theory*

† One direction is obvious; the other comes from the fact that, since the set of cylinders is countable, we can find a sequence φ of density 1 that is a subsequence of φ_u for each u from a certain rank N_u onwards.

Definition 2 (Character) *A* character *of a group $\mathcal{G}$ is a continuous group homomorphism $\mathcal{G} \to \mathbb{T}^1$, where $\mathbb{T}^1$ is the unit circle group (under multiplication). Denote by $\widehat{\mathcal{G}}$ the group of characters of $\mathcal{G}$.*

The following result is well-known (see e.g. **[6]**, Lemma 4.1.3.):

Proposition 2. *Any finite group $\mathcal{G}$ is isomorphic to its dual $\widehat{G}$.*

If $\mathcal{A}$ is a finite abelian group, $\widehat{\mathcal{A}^{\mathbb{Z}}}$ is in bijective correspondance with the sequences of $(\widehat{\mathcal{A}})^{\mathbb{Z}}$ whose elements are all $\mathbf{1}$ except for a finite number. That is, $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ can be written as $\chi(x) = \prod_{k\in\mathbb{Z}} \chi_k(x_k)$ where all but finitely many elements are equal to 1. In this context we call elements of $\widehat{\mathcal{A}}$ *elementary characters.*

Definition 3. *Let $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ and $(\chi_i)_{i\in\mathbb{Z}}$ its decomposition in elementary characters. The* support *of χ is $\mathrm{supp}(\chi) = \{i \in \mathbb{Z} \ : \ \chi_i \neq 1\}$. Its* rank *is $\mathrm{rank}(\chi) = |\,\mathrm{supp}(\chi)|$.*

Definition 4 (Fourier coefficients, or Fourier-Stieltjes transform) *The* Fourier coefficients *of a measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ are given by:*

$$\widehat{\mu}[\chi] = \int_{\mathcal{A}^{\mathbb{Z}}} \chi d\mu$$

for all characters $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$. For a character $\chi = \prod_{k\in S} \chi_k$ (where $S = \mathrm{supp}(\chi)$), this can be rewritten as a finite sum:

$$\widehat{\mu}[\chi] = \sum_{u\in\mathcal{A}^S} \prod_{k\in S} \chi_k(u_k) \cdot \mu([u]_S)$$

Its Fourier coefficients completely characterise μ. What's more, they behave well with regard to convergence in (weak-*) topology:

Theorem 1 (Lévy's continuity theorem) *For $\mu_1, \mu_2, \dots \mu_\infty \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$,*

$$\mu_n \to \mu_\infty \text{ in the weak-* topology} \quad \Longleftrightarrow \quad \forall\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}},\ \widehat{\mu_n}[\chi] \to \widehat{\mu_\infty}[\chi].$$

This theorem was first introduced in **[15]** (in French) ; For a modern proof, see **[2]**, Theorem 26.3.

Definition 5 (Harmonically mixing measure) *$\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is harmonically mixing if, for all $\varepsilon > 0$, there exists $R > 0$ such that $\mathrm{rank}(\chi) > R \implies \widehat{\mu}[\chi] < \varepsilon$.*

Throughout this paper, we sometimes omit to specify the class of initial measures which is always the class of harmonically mixing measures.

Proposition 3. *Let $\mathcal{A}$ be any finite abelian group. Any Bernoulli or (n-step) Markov measure on $\mathcal{A}^{\mathbb{Z}}$ with nonzero parameters is harmonically mixing.*

This is Propositions 6 and 8 and Corollary 10 in **[21]**.

Proposition 4. *For any $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that, if $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ satisfy:*

$$\forall\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}} : \ \mathrm{rank}\,\chi \le r_\varepsilon \implies |\mu[\chi] - \nu[\chi]| \le \frac{1}{r_\varepsilon},$$

then $d(\mu, \nu) \le \varepsilon$.

Proof. The set of characters of rank at most $2k+1$ contains all characters whose support is included in $[-k,k]$, which can be seen as the set of characters for the group $\mathcal{A}^{[-k,k]}$. Applying Lévy's continuity theorem to this group, we get that for $\mu_1, \mu_2, \dots \mu_\infty \in \mathcal{M}(\mathcal{A}^{[-k,k]})$,

$$\mu_n \to \mu_\infty \text{ in the weak-* topology} \iff \forall \chi \in \widehat{\mathcal{A}^{[-k,k]}} : \widehat{\mu_n}[\chi] \to \widehat{\mu_\infty}[\chi].$$

By compactness, this implies that for any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $k_\varepsilon \in \mathbb{N}$ such that, for any pair $\eta_1, \eta_2 \in \mathcal{M}(\mathcal{A}^{[-k,k]})$:

$$\left(\forall \chi \in \widehat{\mathcal{A}^{[-k,k]}}, |\eta_1[\chi] - \eta_2[\chi]| \leq \frac{1}{k_\varepsilon}\right) \Rightarrow \left(\forall u \in \mathcal{A}^{2k+1}, |\eta_1([u]_{-k}) - \eta_2([u]_{-k})| \leq \frac{\varepsilon}{2}\right).$$

In particular, if two measures $\mu, \nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ satisfy $|\mu[\chi] - \nu[\chi]| \leq \frac{1}{k_\varepsilon}$ for any character of rank at most $2k+1$, then $|\mu([u]_{-k}) - \nu([u]_{-k})| \leq \frac{\varepsilon}{2}$ for any $u \in \mathcal{A}^{2k+1}$, which is turn implies:

$$d_{\mathcal{M}}(\mu, \nu) \leq \sum_{i=0}^{k} \frac{\varepsilon}{2^{i+1}} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \leq \frac{\varepsilon}{2} + \frac{1}{2^k}.$$

Therefore, by taking $r_\varepsilon = \max(k_\varepsilon, 2\log(\varepsilon/2) + 1)$, we obtain the desired result. □

3. χ-*diffusivity*

Let $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ and let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be an abelian cellular automaton. Then $\chi \circ F$ is a character (composition of continuous group homomorphisms). One of the central ideas introduced in **[21]** is to focus on the evolution of the rank of characters under the action of F in order to establish randomization in density of harmonically mixing measures. They introduce the following notion of diffusivity over characters. We call it χ-diffusivity to clearly distinguish from the notion of diffusivity we will introduce later.

Definition 6 (Character diffusion) *Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be an abelian cellular automaton. F is strongly* χ-diffusive *(resp. χ-diffusive in density) if, for every nontrivial character $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$,* $\mathrm{rank}(\chi \circ F^t) \to \infty$ *(resp. in density).*

Definition 7. *A measure μ is* strongly nonuniform *if $\mu[\chi] \neq 0$ for all characters $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$.*

For example, a nondegenerate Bernoulli measure $\mu = \otimes\beta$ whose parameters are all equal except one is strongly nonuniform:

Proof. Assume without loss of generality that $\beta(a) = c$ for all $a \in \mathcal{A}$ except for $a = 0$. Let χ_0 be a nontrivial elementary character, i.e. a character of $\mathcal{A}$.

$$\mu[\chi_0] = \sum_{a \in \mathcal{A}} \beta(a)\chi_0(a) = (\beta(0) - c)\chi_0(0) + c\sum_{a \in \mathcal{A}} \chi_0(a) = \beta(0) - c \neq 0$$

by hypothesis. It follows that for every character $\chi = \prod_k \chi_k, \mu[\chi] = \prod_k \mu[\chi_k] \neq 0$ where we use the fact that μ is a Bernoulli measure. □

The following proposition completes Theorem 12 of [**21**] by giving an equivalence between χ-diffusivity and randomization. It also shows that randomization is a structural phenomenon, in the sense that it cannot happen on individual initial measures without happening on a large class.

PROPOSITION 5. *Let F be an abelian cellular automaton. The following are equivalent:*

(i) F is χ-diffusive ;

(ii) F randomizes the class of harmonically mixing measures;

(iii) F randomizes the class of nondegenerate Bernoulli measures;

(iv) F randomizes some strongly nonuniform Bernoulli measure.

This equivalence holds for all three kinds of χ-diffusivity and randomization, that is: strong χ-diffusivity/randomization, χ-diffusivity/randomization in density, and χ-diffusivity for characters of support $\subset \mathbb{U}$/randomization on cylinders of support $\subset \mathbb{U}$ for any $\mathbb{U} \subset \mathbb{Z}$.

Proof.

(i) $\Rightarrow$ *(ii)*: Assume F is strongly χ-diffusive . Let μ be an harmonically mixing measure and χ any nontrivial character of $\mathcal{A}^{\mathbb{Z}}$. Since F is strongly χ-diffusive , $\mathrm{rank}(\chi \circ F^t) \underset{t\to\infty}{\longrightarrow} \infty$. Since μ is harmonically mixing, it follows that $F^t\mu[\chi] = \mu[\chi \circ F^t] \underset{t\to\infty}{\longrightarrow} 0 = \lambda[\chi]$. Since this is true for any character χ, we have by Lévy's continuity theorem $F^t\mu \underset{t\to\infty}{\longrightarrow} \lambda$.

For randomization in density, do the same proof where each convergence is taken along a subsequence of upper density 1.

For randomization for characters of support $\subset \mathbb{U}$, the same argument shows that $F^t\mu[\chi] = \mu[\chi \circ F^t] \underset{t\to\infty}{\longrightarrow} 0 = \lambda[\chi]$ for any nontrivial character χ with support in $\mathbb{U}$. There is a bijection between the characters of support $\subset \mathbb{U}$ and $\widehat{\mathcal{A}^{\mathbb{U}}}$, and we can see a conditional measure $\mu(\cdot \mid \mathfrak{B}_{\mathbb{U}})$ as a measure of $\mathcal{M}(\mathcal{A}^{\mathbb{U}})$. Applying Lévy's continuity theorem to $\mathcal{A}^{\mathbb{U}}$, it follows that $F^t\mu(\cdot \mid \mathfrak{B}_{\mathbb{U}}) \to \lambda(\cdot \mid \mathfrak{B}_{\mathbb{U}})$.

(ii) $\Rightarrow$ *(iii)*: We prove that any nondegenerate Bernoulli measure $\mu = \otimes\beta$ is harmonically mixing. First note that for any elementary character $\chi_0 \neq \mathbf{1}$, we have:

$$\mu[\chi_0] = \int_{\mathcal{A}^{\mathbb{Z}}} \chi d\mu = \sum_{a\in\mathcal{A}} \chi_0(a)\beta(a).$$

Since μ is nondegenerate, at least two values $\beta(a)$ are nonzero. By convexity, since $\chi_0(a)$ are roots of unity, $\mu[\chi_0]$ is a non-extremal point of the unit disk, that is to say $|\mu[\chi_0]| < 1$.

Define $m = \max\{\mu[\chi_0] \ : \ \chi_0 \in \widehat{\mathcal{A}}\backslash\mathbf{1}\} < 1$. For any character $\chi = \prod_{i\in\mathbb{Z}} \chi_i$,

$$|\mu[\chi]| = \left|\prod_{i\in\mathbb{Z}} \mu[\chi_i]\right| \leq m^{\mathrm{rank}(\chi)},$$

where the first equality comes from the fact that μ is a Bernoulli measure. This implies that μ is harmonically mixing.

$(iii) \Rightarrow (iv)$: Trivial.

$(iv) \Rightarrow (i)$: Assume F is not strongly χ-diffusive and take a character $\chi \neq \mathbf{1}$ such that $\mathrm{rank}(F^t \circ \chi) \not\to \infty$. This means that there exists $C \in \mathbb{N}$ and a subsequence φ such that $\mathrm{rank}(\chi \circ F^{\varphi(t)}) \leq C$. Since there is a finite number of such characters, we have for any strongly nonuniform measure μ:

$$|F^{\varphi(t)}\mu[\chi]| \geq \min\{|\mu[\chi]| : \mathrm{rank}(\chi) \leq C\} > 0.$$

Therefore $F^t\mu[\chi] \not\to 0$, which implies that $F^t\mu \not\to \lambda$. We conclude by contraposition.

For randomization in density, do the same proof along a sequence of times with positive upper density.

For randomization of cylinders of support $\subset \mathbb{U}$, do the same proof for any character of support $\subset \mathbb{U}$. □

REMARK 1. *The strongly nonuniform hypothesis is necessary to prevent this kind of counterexample: take $\mathcal{A} = (\mathbb{Z}/2\mathbb{Z})^2$, $F' = F \times Id$ where F is a strongly randomizing cellular automaton on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ (such as F_2, as we prove later) and $\mu = \nu \times \lambda$ where ν is any harmonically mixing measure and λ is the uniform Bernoulli measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$. Then F' strongly randomizes μ, but does not strongly randomize any measure whose second component is nonuniform.*

LEMMA 1. *Let F be an abelian CA of neighborhood $\mathcal{N}$. Then for any character χ,* $\mathrm{rank}(\chi \circ F) \leq |\mathcal{N}| \cdot \mathrm{rank}(\chi)$.

Proof. Let F be an abelian cellular automaton. It can be written as $F = \sum_{k\in\mathcal{N}} \phi_k \circ \sigma_k$ where $\mathcal{N} \subseteq \mathbb{Z}$ is a finite set and ϕ_k are endomorphisms of $(\mathcal{A}, +)$.

For any $\chi \in \widehat{\mathcal{A}^{\mathbb{Z}}}$ and $x \in \mathcal{A}^{\mathbb{Z}}$, we then rewrite:

$$\begin{aligned}\chi \circ F(x) &= \prod_{i\in\mathbb{Z}} \chi_i \left(\sum_{k\in\mathcal{N}} \phi_k(x_{i+k})\right) \\ &= \prod_{i\in\mathbb{Z}} \prod_{k\in\mathcal{N}} \chi_i \circ \phi_k(x_{i+k}) \\ &= \prod_{i\in\mathbb{Z}} \left(\prod_{k\in\mathcal{N}} \chi_{i-k} \circ \phi_k\right)(x_i)\end{aligned}$$

where the last step is by rewriting the sum $i \mapsto k - i$. Therefore:

$$(\chi\circ F)_i \neq \mathbf{1} \iff \prod_{k\in\mathcal{N}} \chi_{i-k}\circ\phi_k \neq 1 \Longrightarrow \exists k \in \mathcal{N},\ \chi_{i-k}\circ\phi_k \neq 1 \Longrightarrow \exists k \in \mathcal{N},\ \chi_{i-k} \neq 1.$$

It follows that $\mathrm{rank}(\chi \circ F) \leq |\mathcal{N}| \cdot \mathrm{rank}(\chi)$. □

The converse lemma holds for reversible CA:

LEMMA 2. *Let F be a reversible abelian CA. There exists a constant C such that* $\text{rank}(\chi \circ F) \geq C \cdot \text{rank}(\chi)$.

DEFINITION 8 (DEPENDENCY FUNCTION) *To any abelian CA $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ we associate a* dependency function*:*

$$\forall (t,i) \in \mathbb{N} \times \mathbb{Z},\ \Delta_F(t,i) = q \in \mathcal{A} \mapsto F^t(x_q)_i$$

where x_q is the configuration worth q at position 0 and 0 everywhere else.

Notice that by linearity, we have:

$$\forall x \in \mathcal{A}^{\mathbb{Z}},\ \forall t \in \mathbb{N},\ F^t(x)_z = \sum_{j \in \mathbb{Z}} \Delta_F(t, z-j)(x_j), \tag{1}$$

where only a finite number of terms are nonzero.

In the following lemma, we prove that the support of the image of a fixed character at time t is entirely determined by the local dependency diagram.

LEMMA 3. *Let F be an abelian cellular automaton, and χ be a character whose support is included in $[0,m]$ for $m \geq 0$. If there are (t_1, z_1) and (t_2, z_2) such that:*

$$\forall z \in [0,m],\ \Delta_F(t_1, z_1 + z) = \Delta_F(t_2, z_2 + z)$$

then

$$-z_1 \in \text{supp}(\chi \circ F^{t_1}) \Leftrightarrow -z_2 \in \text{supp}(\chi \circ F^{t_2})$$

Proof. By Equation 1,

$$\begin{aligned}
\chi \circ F^t(x) &= \prod_{i \in \mathbb{Z}} \chi_i \left(\sum_{j \in \mathbb{Z}} \Delta_F(t, i-j)(x_j) \right) \\
&= \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \chi_i \circ \Delta_F(t, i-j)(x_j) \\
&= \prod_{j \in \mathbb{Z}} \left(\prod_{k \in \mathbb{Z}} \chi_{k+j} \circ \Delta_F(t,k) \right)(x_j)
\end{aligned}$$

where the last step is by rewriting the sum: $k = i - j$. It follows:

$$\begin{aligned}
-z_1 \in \text{supp}(\chi \circ F^{t_1}) &\Leftrightarrow \prod_{k \in \mathbb{Z}} \chi_{k - z_1} \circ \Delta_F(t_1, k) \neq 0 \\
&\Leftrightarrow \prod_{k \in \mathbb{Z}} \chi_{k - z_2} \circ \Delta_F(t_2, k) \neq 0 \\
&\Leftrightarrow -z_2 \in \text{supp}(\chi \circ F^{t_2})
\end{aligned}$$

where the second step uses the hypothesis of the lemma and the fact that $\chi_{k-z_1} = 0$ whenever $k - z_1 \notin [0,m]$. □

Given an abelian CA F and $t \in \mathbb{N}$, denote by $d(t)$ the number of non-trivial dependencies of F at time t:

$$d(t) = \left|\{z \in \mathbb{Z} : \Delta_F(t,z) \neq 0\}\right| \quad \text{(the zero map)}$$

Following **[21, 22]**, we introduce *isolated bijective dependencies* which provide useful lower-bounds on the rank of image of characters under the action of F.

Definition 9 (Isolated dependency) *For $k \geq 1$, a k-isolated dependency is a pair $(t,z) \in \mathbb{N} \times \mathbb{Z}$ such that:*

1. *$\Delta_F(t,z)$ is a bijective dependency;*
2. *$\Delta_F(t,z+i) = 0$ for $1 < i \leq k$.*

We denote by $S_k(t)$ the set of k-isolated dependencies at time t (i.e. of the form (t,z)) and $s_k(t) = |S_k(t)|$.

This concept of k-isolated dependencies is also used in **[23]** to define dispersion mixing measure and dispersive CA. The main techniques of **[21]** (proof of Therorem 15) and **[22]** (V-separating sets) is essentially to use s_k as a lower bound for the rank of characters under the iteration of an abelian CA.

Proposition 6. *Let F be an abelian cellular automaton, $\chi \neq \mathbf{1}$ a character, and k the diameter of* $\operatorname{supp}(\chi)$*. Then we have:*

$$\forall t \in \mathbb{N},\ s_{k-1}(t) \leq \operatorname{rank}(\chi \circ F^t) \leq \operatorname{rank}(\chi) \cdot d(t)$$

Proof. The rank being invariant by translation, we can suppose that $\operatorname{supp}(\chi) \subset [0, k-1]$ and that $\chi_0 \neq 1$. By Equation 1, we have:

$$\chi \circ F^t(x) = \prod_{i=0}^{k-1} \prod_{j \in \mathbb{Z}} \chi_i \circ \Delta_F(t, i-j)(x_j) = \prod_{j \in \mathbb{Z}} \left(\prod_{i=0}^{k-1} \chi_i \circ \Delta_F(t, i-j) \right)(x_j).$$

First, we have

$$j \in \operatorname{supp}(\chi \circ F^t) \Rightarrow \exists i, \chi_i \circ \Delta_F(t, i-j) \neq 1 \Rightarrow \exists i \in \operatorname{supp}(\chi), \Delta_F(t, i-j) \neq 0.$$

Therefore we get the upper bound $\operatorname{rank}(\chi \circ F^t) \leq \operatorname{rank}(\chi) \cdot d(t)$.

Second, if $(t, -j)$ is $k-1$-isolated, then $j \in \operatorname{supp}(\chi \circ F^t)$. Indeed:

$$(\chi \circ F^t)_j = \prod_{i=0}^{k-1} \chi_i \circ \Delta_F(t, i-j) = \chi_0 \circ \Delta_F(t, -j).$$

We deduce that $s_{k-1}(t) \leq \operatorname{rank}(\chi \circ F^t)$. □

Example 1. *By Propositions 5 and 6, having $s_k(t) \longrightarrow_t +\infty$ for all k is a sufficient condition for randomization, but it is not necessary. For instance, take $F' = F \times (\sigma^N \circ F)$ where F is any abelian CA that randomizes in density. By Corollary 1, F' is randomizing in density. However, when N is large enough that*

F and $\sigma^N \circ F$ have disjoint neighborhoods, F' has no bijective dependency, and therefore $s_k(t) = 0$ for all k and t.

On the other hand, having many bijective dependencies is not enough if they are not well-isolated. For example, one can check that H_2 satisfies $s_1(t) = t - 2$ but it is not randomizing, as shown in Section 5.

4. *Duality, Diffusivity and Solitons*

In the last section, we saw that the iterated images of characters under the action of a CA is key to understanding its action on probability measures. It turns out that the action of abelian CA on characters can be seen as a CA on the dual group, and that furthermore this dual CA shares many properties with the original CA.

Remember that any character χ of $\widehat{G^{\mathbb{Z}}}$ can be written as a finite product of cellwise (elementary) characters: $\chi(x) = \prod_{i\in\mathcal{N}} \chi_i(x_i)$ for some finite set $\mathcal{N} \subset \mathbb{Z}$ and $\chi_z \in \widehat{G}$. To such a χ we associate $\Psi(\chi)$ the configuration of $\widehat{G}^{\mathbb{Z}}$ defined by:

$$\Psi(\chi)(z) = \begin{cases} \chi_z & \text{if } z \in A \\ 1 & \text{else.} \end{cases}$$

Note that $\Psi\left(\widehat{G^{\mathbb{Z}}}\right)$ is exactly the set of finite configurations of $\widehat{G}^{\mathbb{Z}}$.

Definition 10 (Dual CA) *Let F be an abelian CA over $G^{\mathbb{Z}}$. It can be written as*

$$F(x)_z = \sum_{i\in\mathcal{N}} \phi_i(x_{z+i})$$

where $\mathcal{N} \subset \mathbb{Z}$ is finite and ϕ_i are endomorphisms of G. We define $\widehat{F}$ over the finite configurations of $\widehat{G}^{\mathbb{Z}}$ by:

$$\widehat{F}(\Psi(\chi)) = \Psi(\chi \circ F).$$

Since $\widehat{F}$ is continuous and shift-invariant on finite configurations, it can be extended by continuity to a cellular automaton $\widehat{G}^{\mathbb{Z}} \to \widehat{G}^{\mathbb{Z}}$; this is the dual CA of F, and is an abelian CA for the group $(\widehat{G}, \times)$.

More concretely, if $\chi(x) = \prod_{z\in A} \chi_z(x_z)$, we have:

$$\widehat{F}(\Psi(\chi)) = \begin{cases} x_z \mapsto \prod_{i\in\mathcal{N}} \chi_{z-i}\big(\phi_i(x_z)\big) & \text{if } z \in A + \mathcal{N} \\ 1 & \text{else.} \end{cases}$$

Then, for any $c \in \widehat{G}^{\mathbb{Z}}$ we can define:

$$\widehat{F}(c)_z = \prod_{i\in V} \gamma_i(c_{z-i}) \tag{2}$$

where γ_i is the endomorphism of $\widehat{G}$ defined by:

$$\gamma_i(\chi) = g \mapsto \chi \circ \phi_i(g).$$

When $G = \mathbb{F}_p^d$, the dual of a CA is obtained (up to conjugacy) by applying a mirror operation and transposing the matrix corresponding to each coefficient. Indeed, the map:

$$\begin{array}{ccc} G & \to & \widehat{G} \\ a & \mapsto & \chi_a \end{array} \qquad \text{where} \qquad \chi_a : b \in \mathcal{A} \mapsto e^{\frac{2i\pi}{p} <a,b>},$$

where $< a, b >$ denotes the scalar product of a and b seen as d-dimensional vectors, is an isomorphism. Through that isomorphism, we have that $\chi_a \circ M = \chi_{M^t a}$ for any endomorphism $M : \mathcal{A} \to \mathcal{A}$, and the result comes from Equation (2).

In particular, our examples F_2 and H_2 are flip conjugate to their own dual since all their coefficients are symmetric matrices. H_2 is actually conjugate to its dual since it is left-right symmetric.

We do not know whether an abelian CA F is always flip conjugate to its dual $\widehat{F}$; however, we show in the remainder of this section that they are dynamically close enough that properties like randomization or diffusion are preserved by duality.

LEMMA 4. *Let Φ_1 and Φ_2 be two abelian CA over $G^{\mathbb{Z}}$. Then we have $\widehat{\Phi_1 \circ \Phi_2} = \widehat{\Phi_2} \circ \widehat{\Phi_1}$. As a consequence:*

- $\widehat{F^t} = \left(\widehat{F}\right)^t$ *for any $t > 0$,*
- $\widehat{F \circ \sigma} = \widehat{F} \circ \sigma^{-1}$,
- *If F is reversible, then $\widehat{F}$ is also reversible and $\widehat{F^{-1}} = \left(\widehat{F}\right)^{-1}$.*

Furthermore, $\widehat{\widehat{F}} = F$ up to a canonical isomorphism.

Proof. By definition of dual CAs, we have for any $\chi \in \widehat{G^{\mathbb{Z}}}$:

$$\widehat{\Phi_1 \circ \Phi_2}(\Psi(\chi)) = \Psi(\chi \circ \Phi_1 \circ \Phi_2) = \widehat{\Phi_2}(\Psi(\chi \circ \Phi_1)) = \widehat{\Phi_2} \circ \widehat{\Phi_1}(\Psi(\chi)).$$

Since $\Psi\left(\widehat{G^{\mathbb{Z}}}\right)$ is dense in $\widehat{G}^{\mathbb{Z}}$ we deduce that $\widehat{\Phi_1 \circ \Phi_2} = \widehat{\Phi_2} \circ \widehat{\Phi_1}$ on the whole space.

For the last point, it is well-known that $\widehat{\widehat{G}} \simeq G$ through the canonical isomorphism $\psi : g \mapsto (\chi \mapsto \chi(g))$ (see e.g.[**6**], Lemma 4.1.4). Then we check that $\widehat{\widehat{F}} : \psi(g) \mapsto \psi(g) \circ \widehat{F} = \psi(F(g))$, so that $\widehat{\widehat{F}} \simeq F$ up to this isomorphism. □

In the following definition and in the remainder of the section, we stress when our results do not require the CA to be abelian, even though we will only apply them to abelian CA.

DEFINITION 11. *A CA F with a quiescent state 0 is* strongly diffusive *(resp. diffusive in density) if for any finite configuration c, we have $\text{rank}\big(F^t(c)\big) \to \infty$ (resp. on a subsequence of density 1).*

LEMMA 5. *An abelian CA F is strongly χ-diffusive (resp. χ-diffusive in density) if and only if $\widehat{F}$ is strongly diffusive (resp. diffusive in density).*

Proof. $\widehat{F}(\Psi(\chi)) = \Psi(\chi \circ F)$ and $\text{rank}(\chi) = \text{rank}\big(\Psi(\chi)\big)$. □

Definition 12 (Soliton) *Let F be an abelian CA. A* soliton *is a finite configuration $c \neq \overline{0}$ such that $F^p(c) = \sigma^q(c)$ for some $p \in \mathbb{N}$ and $q \in \mathbb{Z}$.*

Intuitively, having a soliton is contradictory to being diffusive (even in density). In the remainder of the section we will develop this intuition and prove a series of technical results about solitons that will be culminate in the characterization of randomization in density in the next section.

First note that all the configurations in the orbit of a soliton have bounded rank. Conversely, we can extract a soliton from any orbit of finite configurations whose rank is bounded on a set of time steps of positive density:

Proposition 7. *Let $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a surjective cellular automaton with a quiescent state 0. Assume that F is not diffusive in density. Then F admits a soliton.*

Proof. There is a finite initial configuration x and an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of positive upper density such that $\mathrm{rank}(F^{T_n}(x))$ is bounded. Without loss of generality, assume that $\mathrm{rank}(F^{T_n}(x)) = k$ for all n. Denote $i_1(T_n), \ldots, i_k(T_n)$ the nonzero coordinates at time T_n.

Now let m be the maximum integer such that there exists an integer M such that the subsequence:

$$(T_{\varphi(n)})_{n\in\mathbb{N}} = \{n \in \mathbb{N} : i_m(T_n) - i_1(T_n) \leq M\}$$

has positive upper density. In particular $(T_{\varphi(n)})_{n\in\mathbb{N}}$ has positive upper density. We distinguish two cases.

$\boxed{m = k:}$ $F^{T_{\varphi(n)}}(x)_{[i_1(T_{\varphi(n)}), i_k(T_{\varphi(n)})]}$ can take at most $|\bigcup_{j=0}^{M} \mathcal{A}^j| = \sum_{j=0}^{M} |\mathcal{A}|^j$ different values. By the pidgeonhole principle, we can find two integers $a < b$ such that

$$F^{T_{\varphi(a)}}(x)_{[i_1(T_{\varphi(a)}), i_k(T_{\varphi(a)})]} = F^{T_{\varphi(b)}}(x)_{[i_1(T_{\varphi(b)}), i_k(T_{\varphi(b)})]}.$$

But this means that $F^{T_{\varphi(b)}}(x) = \sigma^{T_{\varphi(b)} - T_{\varphi(a)}} \circ F^{T_{\varphi(a)}}(x)$, so we found a soliton.

$\boxed{m < k:}$ First, by the same argument as above, $F^{T_{\varphi(n)}}(x)_{[i_1(T_{\varphi(n)}), i_m(T_{\varphi(n)})]}$ can only take a finite number of values, so at least one of these words appear with positive density. Denote w the corresponding word and $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ the corresponding subsequence. We now prove that the configuration

$$\cdots 0 \cdot 0 \cdot 0 \cdot w \cdot 0 \cdot 0 \cdot 0 \cdots$$

is a soliton.

Take $N \in \mathbb{N}$ such that $\frac{1}{N}$ is a lower bound on the density of $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ and let r be the radius of F. By construction of m, times $T_{\varphi'(n)}$ where $i_{m+1}(T_{\varphi'(n)}) - i_m(T_{\varphi'(n)}) \leq 2rN$ have upper density 0. Therefore we extract from the sequence $(T_{\varphi'(n)})_{n\in\mathbb{N}}$ a new subsequence $(T_{\varphi''(n)})_{n\in\mathbb{N}}$ corresponding to times t where $i_{m+1}(t) - i_m(t) > 2rN$ with the same upper density. In particular, we can find some n such that $T = T_{\varphi''(n+1)} - T_{\varphi''(n)} \leq N$.

As shown in Figure 1, only two disjoint areas can contain nonzero values in $F^{T_{\varphi''(n+1)}}(x)$:

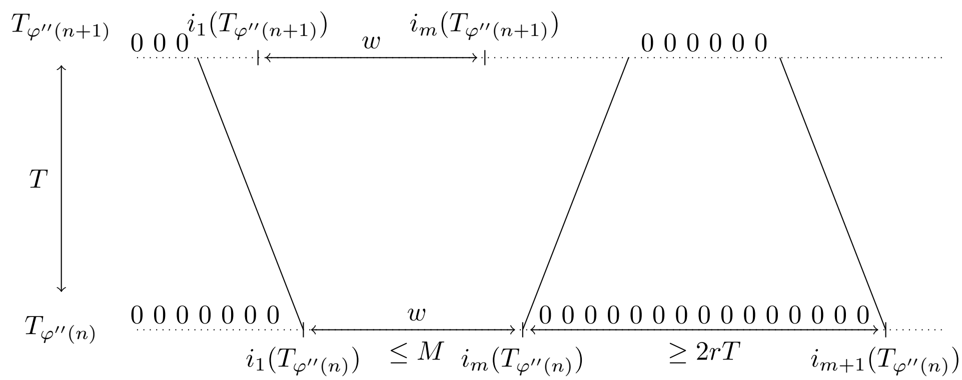

Figure 1. The finite configuration defined by w is a soliton.

- the interval $[i_1(T_{\varphi''(n)}) - rT, i_m(T_{\varphi''(n)}) + rT]$,
  which contains $f^T\left(0^{2rT} \cdot w \cdot 0^{2rT}\right)$;
- the interval $[i_{m+1}(T_{\varphi''(n)}) - rT, i_k(T_{\varphi''(n)}) + rT]$
  which contains $f^T\left(0^{2rT} \cdot F^{T_{\varphi''(n)}}(x)_{[i_{m+1}(T_{\varphi''(n)}), i_k(T_{\varphi''(n)})]} \cdot 0^{2rT}\right)$.

Indeed, any cell outside of these regions can be written as $f^T(0^{2rT}) = 0$ since 0 is quiescent. Consider the different possibilities for the value of $i_1(T_{\varphi''(n+1)})$:

- if $i_1(T_{\varphi''(n+1)}) < i_1(T_{\varphi''(n)}) - rT$, then $F^{T_{\varphi''(n+1)}}(x)_{i_1(T_{\varphi''(n+1)})} = f^T(0^{2rT} = 0$ which is a contradiction;
- if $i_1(T_{\varphi''(n+1)}) > i_1(T_{\varphi''(n)}) + rT$, then $F^{T_{\varphi''(n)}}(x)_{[i_1(T_{\varphi''(n)}) + rT]}$ is a nonzero word whose image under f^T is zero, a contradiction.

Therefore we have $i_1(T_{\varphi''(n+1)}) \in [i_1(T_{\varphi''(n)}) \pm rT]$. In particular, the interval $I = [i_1(T_{\varphi''(n)}) - rT, i_m(T_{\varphi''(n)}) + rT]$ contains all $i_\ell(T_{\varphi''(n+1)})$ for $\ell \leq m$. Using a similar argument, $i_{m+1}(T_{\varphi''(n+1)}) \in [i_m(T_{\varphi''(n)}) \pm rT]$, so that I does not contain any $i_\ell(T_{\varphi''(n+1)})$ for $\ell > m$.

From this we conclude that for some constant C:

$$F^T\left(0^{2rT} \cdot w \cdot 0^{2rT}\right) = 0^C \cdot w \cdot 0^{2rT-C}$$

and therefore we have found a soliton. □

The remainder of this section is dedicated to proving the following proposition.

Proposition 8. *Let F be an abelian CA. F has a soliton if and only if $\widehat{F}$ has a soliton.*

To prove this proposition, we need a series of lemmas. For an abelian CA F, a finite fixed point is just a fixed point which is also a finite configuration. It is non-trivial if it is not the configuration everywhere equal to 0.

LEMMA 6. *Let F be an abelian CA and denote by $X_{F,n}$ the set of fixed points of (spatial) period n: $X_{F,n} = \{x : \sigma^n(x) = x \text{ and } F(x) = x\}$. F has a non-trivial finite fixed point if and only if $n \mapsto |X_{F,n}|$ is unbounded (and in this case we actually have $|X_{F,n}| = \Omega(2^n)$).*

Proof. First suppose that $F(x) = x$ where x is a nontrivial finite configuration and denote by u a finite word containing the non-zero part of x. Let r be the radius of F and $k = |u| + 2r$. Consider the set of finite words :

$$W_n = \left\{ w \in \mathcal{A}^n \ : \ \begin{array}{ll} & \forall i, 0 \le i < \left\lfloor \frac{n}{k} \right\rfloor \Rightarrow \ w_{[i,i+k]} = 0^k \text{ or } w_{[i,i+k]} = 0^r u 0^r \\ \text{and} & \\ & w_{k \cdot \left\lfloor \frac{n}{k} \right\rfloor + 1, n} = 0 \end{array} \right\}.$$

Any periodic configuration whose period is in W_n (excepting 0^n) is a nontrivial fixed point and $|W_n| = 2^{\left\lfloor \frac{n}{k} \right\rfloor}$. Therefore we have $|X_{F,n}| = \Omega(2^n)$.

Conversely, if n and $|X_{F,n}|$ are large enough, then $X_{F,n}$ must contain at least 2 distinct configurations x_1 and x_2 such that $x_1|_{[1,r]} = x_2|_{[1,r]}$ and $x_1|_{[n-r+1,n]} = x_2|_{[n-r+1,n]}$ by the pidgeonhole principle. It follows that the configuration

$$x : z \in \mathbb{Z} \mapsto \begin{cases} 0 & \text{if } z \le 0 \text{ or } z > n \\ x_1(z) - x_2(z) & \text{else} \end{cases}$$

is a non-trivial finite fixed point. □

LEMMA 7. *Let G be an abelian group. For any endomorphism $h : G \to G$, define its dual $\widehat{h}$ by $\widehat{h}(\chi) = \chi \circ h$ for any character $\chi \in \widehat{G}$. Then we have $|\ker(h)| = |\ker(\widehat{h})|$.*

Proof. We have $\chi \in \ker(\widehat{h}) \Leftrightarrow \chi \circ h = 1 \Leftrightarrow \mathrm{Im}(h) \subset \ker(\chi)$. Therefore the restriction

$$\chi \mapsto \chi|_{(G/\operatorname{Im} h)}$$

is a bijection between $\ker(\widehat{h})$ and $(\widehat{G/\operatorname{Im} h})$. Since $\left|(\widehat{G/\operatorname{Im} h})\right| = |G/\operatorname{Im} h| = |\ker h|$ by Proposition 2, we conclude. □

LEMMA 8. *Let F be an abelian CA over alphabet G. F has a non-trivial finite fixed point if and only if $\widehat{F}$ has a non-trivial finite fixed point.*

Proof. For any $n \in \mathbb{N}$, let h_n be the endomorphism of G^n defined as follows. For any $u \in G^n$, let $x^u \in G^{\mathbb{Z}}$ be the (spatially) periodic point of period u, i.e. $x^u_i = u_{i \bmod n}$ for all i.

$$\forall u \in G^n, h_n(u) = (F(x^u) - x^u)_{[0,n-1]}$$

h_n captures the action of the CA $F - Id$ on spatially periodic configurations of period n. In particular we have $|\ker(h_n)| = |X_{F,n}|$. Now consider its dual $\widehat{h_n}$, which is an endomorphism of $\widehat{G^n} = (\widehat{G})^n$: for any $\chi = \prod_{1 \le i \le n} \chi_i$,

$$\begin{aligned}\widehat{h_n}(\chi) : u \mapsto \prod_{i=1}^{n} \chi_i(F(x^u) - x^u)_i &= \prod_{i=1}^{n} \chi_i \circ F(x^u)_i \cdot \chi_i(x^u)_i^{-1} \\ &= (\widehat{F}(x^\chi) \cdot x^{(\chi^{-1})})(x^u)\end{aligned}$$

by definition of $\widehat{F}$, and where again we define the periodic point $x^\chi \in \widehat{G}^{\mathbb{Z}}$ by $x_i^\chi = \chi_{i \mod n}$. Therefore we have $|\ker(\widehat{h_n})| = |X_{\widehat{F},n}|$ similarly as above.

Now, by Lemma 7, we have $|\ker(h_n)| = |\ker(\widehat{h_n})|$ and therefore $|X_{F,n}| = |X_{\widehat{F},n}|$. We conclude by Lemma 6. □

Proof (of Proposition 8). Suppose F has a soliton x, that is, $F^p(x) = \sigma^q(x)$ for $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then x is a finite fixed point for the abelian CA $\sigma^{-q} \circ F^p$. By Lemma 8 we deduce that $\widehat{\sigma^{-q} \circ F^p}$ also has a finite fixed point y. Using Lemma 4, we can rewrite this $(\widehat{F})^p(y) = \sigma^{-q}(y)$, which shows that y is actually a soliton of $\widehat{F}$.

In the same way we prove that a soliton in $\widehat{F}$ implies a soliton in F. □

EXAMPLE 2. *Note that the smallest solitons of F and $\widehat{F}$ need not be of the same size. For example, consider the CA F defined over the alphabet $\mathbb{F}_2^2$ defined by:*

$$F(x)_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x_z + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot x_{z+1}.$$

F acts like the identity on any configuration whose second components are all 0. In particular it admits solitons of rank 1. However, its dual, obtained up to conjugacy by mirroring and transposing the coefficients:

$$\widehat{F}(x)_z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot x_{z-1} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot x_z$$

has no soliton of rank 1. Indeed, take any finite configuration x supported in $[i,j]$ for $i \leq u$ and such that $x_i \neq 0, x_j \neq 0$. Notice that:

$$F(x)_i = \begin{pmatrix} (x_i)_1 \\ (x_i)_1 + (x_i)_2 \end{pmatrix} \neq 0 \qquad \text{and} \qquad F(x)_{j+1} = \begin{pmatrix} 0 \\ (x_j)_2 \end{pmatrix}.$$

In particular, if $(x_j)_2 \neq 0$, then x cannot be a soliton.

Now take any finite x of rank 1, assuming $x_0 \neq 0$. If $(x_0)_2 \neq 0$ then x is not a soliton by the previous argument. If $(x_0)_2 = 0$ then $\widehat{F}(x)$ is a finite configuration of rank 1 such that $\widehat{F}(x)_0 = \begin{pmatrix} (x_0)_1 \\ (x_0)_1 \end{pmatrix}$ and is not a soliton.

Finally, it is easy to check that the configuration

$$y = \cdots \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdots$$

is a fixed point, and therefore a soliton, of $\widehat{F}$.

5. *Characterization of Randomization in Density*

We are now ready to prove our main result regarding randomization in density, which is a combinatorial characterization by the absence of solitons.

THEOREM 2. *Let F be an abelian CA. The following are equivalent:*

(i) F randomizes in density any harmonically mixing measure;

(ii) F has no soliton;

(iii) F is diffusive in density;

(iv) For some strongly nonuniform measure μ, the sequence $(F^t\mu)_{t\in\mathbb{N}}$ admits λ as an accumulation point.

Proof. The equivalence $(i) \Leftrightarrow (iii)$ comes from Propositions 5 and 8, the implication $(ii) \Rightarrow (iii)$ is Proposition 7 and its converse is obvious. Moreover, it is clear that $(i) \Rightarrow (iv)$. We prove $(iv) \Rightarrow (ii)$. Let us suppose (iv) holds. Then for any non-trivial character χ we can make $F^t\mu[\chi] = \mu[\widehat{F}^t(\chi)]$ arbitrarily close to $0 = \lambda[\chi]$ by an appropriate choice of t. However, as shown in the proof of Proposition 5, the fact that μ is strongly nonuniform implies that there is a constant m such that, for any non-trivial character χ, $\mu[\chi] \geq m^{\mathrm{rank}(\chi)}$. Therefore $\mathrm{rank}(\widehat{F}^t(\chi))$ is not bounded from above, which means that $\widehat{F}$ has no soliton. By Proposition 8, we conclude. □

Before giving some general consequences of this theorem and applying it to the commutative case, let's use it on our examples.

EXAMPLE 3. *The CA H_2 admits a soliton:*

$$\cdots \begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\cdots$$

and therefore it is not randomizing in density.

*On the contrary we show that F_2 has no soliton and therefore is randomizing in density. This is an alternative proof to the result of [**17**] where it was proven through a delicate analysis of F_2 using binomial coefficients and Lucas' Lemma. Indeed, it is easy to show by induction that:*

$$\forall c \in \mathcal{A}^{\mathbb{Z}}, \forall n \in \mathbb{N}, \forall z \in \mathbb{Z},\ F_2^{2^{n+1}}(c)_z + F_2^{2^n}(c)_{z+2^n} + c_z = 0 \bmod 2.$$

Suppose by contradiction that F_2 has a soliton c such that $F^p(c) = \sigma^q(c)$. Therefore there is a constant M such that any non-zero cell of the space-time diagram $(F^t(c))_{t\in\mathbb{N}}$ is at distance at most M of the real line $L = \{(z,t) : pz - qt = 0\}$. In other words, $F^t(c)_z \neq 0 \Rightarrow |pz - qt| < pM$. Now take n such that $2^n > M$ and any $|z| \leq M$ and distinguish three cases:

$q = 0$ *$c_{z-2^n} = 0$ and $F_2^{2^{n+1}}(c)_{z-2^n} = 0$, so that $F_2^{2^n}(c)_z = 0$. This is true for every point at distance at most M from the line at time 2^n, so $F_2^{2^n}(c) = 0$, a contradiction.*

$q = p$ $c_{z+2^{n+1}} = 0$ *and* $F_2^{2^n}(c)_{z+2^{n+1}+2^n} = 0$*, so that* $F_2^{2^{n+1}}(c)_{z+2^{n+1}} = 0$*. This is true for every* z *at distance at most* M *from the line at time* 2^{n+1}*, so* $F_2^{2^{n+1}}(c) = 0$*, a contradiction.*

otherwise, $F_2^{2^{n+1}}(c)_z = F_2^{2^n}(c)_{z+2^n} = 0$ *when* n *is large enough, so that* $c_z = 0$ *for all* $|z| \leq M$*, a contradiction.*

TO DO: Maybe here a figure

DEFINITION 13 (POSITIVE EXPANSIVENESS) *A CA is* positively expansive *if there is some finite* $W \subseteq \mathbb{Z}$ *such that for any pair of distinct configurations* $x, y \in \mathcal{A}^{\mathbb{Z}}$*:*

$$\exists t \in \mathbb{N},\ \exists z \in W,\ F^t(x)_z \neq F^t(y)_z.$$

More generally, for $\alpha \in \mathbb{R}$*, we say that* F *is* positively expansive in direction α *if there is some finite* $W \subseteq \mathbb{Z}$ *such that for any pair of distinct configurations* $x, y \in \mathcal{A}^{\mathbb{Z}}$*,*

$$\exists t \in \mathbb{N},\ \exists z \in W,\ F^t(x)_{z+\lceil \alpha t \rceil} \neq F^t(y)_{z+\lceil \alpha t \rceil}$$

See [**24, 8**] for further developments on directional dynamics in cellular automata.

Recall also that a *subautomaton* F' of an abelian CA F over group G is an abelian CA induced by the restriction of F to a stable subgroup G' of G: $F' = F_{|G'^{\mathbb{Z}}}$.

COROLLARY 1. *Let* F *and* G *be abelian CA. It holds:*

- *if* F *and* G *are randomizing then so does* $F \times G$*;*
- *if* F *is randomizing then all its subautomata are;*
- *if* F *is randomizing and reversible, then so is* F^{-1}*;*
- *if* F *has a direction of positive expansivity then it is randomizing;*

where randomizing means randomizing in density any harmonically mixing measure.

Proof. This corollary follows from Theorem 2 by the following elementary observations on solitons:

- a soliton in $F \times G$ implies a soliton in either F or G;
- a soliton in a subautomaton of F is a soliton for F;
- a soliton for F^{-1} is a soliton for F;
- a positively expansive CA cannot admit a soliton. Indeed, take a CA F with a direction of positive expansiveness α and assume for the sake a contradiction that it admits a soliton c: $F^p(c) = \sigma^q(c)$, such that $\frac{q}{p} > \alpha$. Then, for any finite $W \subset \mathbb{Z}$, we take two distinct configurations $x = 0$ and $y = \sigma^n(c)$, and we can check that for n large enough $F^t(x)_{z+\lceil \alpha t \rceil} \neq F^t(y)_{z+\lceil \alpha t \rceil}$ for every $t \in \mathbb{N}$ and $z \in W$.

□

Remark 2. *A CA with local rule $f : \mathcal{A}^m \to \mathcal{A}$ is bipermutive if the maps $x \mapsto f(x, a_1, \ldots, a_{m-1})$ and $x \mapsto f(a_1, \ldots, a_{m-1}, x)$ are permutations of $\mathcal{A}$ for any $a_1, \ldots, a_{m-1} \in \mathcal{A}$. Since bipermutivity implies the existence of a direction of positive expansivity [4], the above corollary implies that any bipermutive abelian CA randomizes in density any harmonically mixing measure. This generalizes Theorem 9 of [22], where the authors consider abelian CA of the form*

$$F = \sum_{i \in \mathcal{N}} \overline{\phi}_i \circ \sigma^i$$

where $|\mathcal{N}| \geq 2$ and $\overline{\phi}_i$ are commuting automorphisms. We do not need this hypothesis here, and for instance we prove that the following CA over $\mathbb{F}_2^2$ is randomizing in density:

$$F(c)_z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot c_{z+1}.$$

6. *Other forms of randomization*

In this section we consider other forms of randomization that have been less studied in the literature. First we prove that, in the case of abelian CA whose coefficients are commuting endomorphisms, only randomization in density can happen. Then we provide examples of abelian CA that exhibit strong randomization and randomization for cylinders up to some fixed length.

6.1. *Abelian CAs with commuting coefficients*

The case of abelian CA with commuting coefficients is in many regards similar to the case of scalar coefficients. These CA have more rigidity in their time evolution than general abelian CA: the image of a single cell at time t can be determined directly through the use of binomial theorem and modular arithmetic of binomial coefficients. In particular, when t is some power the order of the group, the number of bijective dependencies is bounded, which explains why these CA cannot randomize strongly.

Lemma 9. *Let p be a prime number and $l \geq 0$. Let $(\mathcal{X}, +, \times)$ be a commutative ring of characteristic p^l, i.e. such that for any $X \in \mathcal{X}$:*

$$p^l \cdot X = 0$$

where 0 is the neutral element for $+$.

Then, for any n and any elements $X_i \in \mathcal{X}$, $1 \leq i \leq k$, we have this equality in $\mathcal{X}$:

$$\left(\sum_{i=1}^{k} X_i\right)^{p^{n+l-1}} = \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}}$$

Proof. First by the binomial theorem we have:

$$\left(\sum_{i=1}^{k} X_i\right)^{p^n} = \left(X_1 + \sum_{i=2}^{k} X_i\right)^{p^n} = X_1^{p^n} + \left(\sum_{i=2}^{k} X_i\right)^{p^n} + p \cdot Y$$

for some Y because p divides $\binom{p^n}{i}$ for $0 < i < p^n$. By a direct induction we deduce:

$$\left(\sum_{i=1}^{k} X_i\right)^{p^n} = \sum_{i=1}^{k} X_i^{p^n} + p \cdot Y'$$

for some Y'. Now, applying again the binomial theorem we get:

$$\begin{aligned}\left(\sum_{i=1}^{k} X_i\right)^{p^{n+1}} &= \left(\sum_{i=1}^{k} X_i^{p^n} + p \cdot Y'\right)^{p} \\ &= \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p} + \sum_{j=1}^{p} \binom{p}{j} \cdot p^j \cdot (Y')^j \times \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p-j} \\ &= \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p} + p^2 \cdot Z\end{aligned}$$

because p^2 divides $\binom{p}{j} \cdot p^j$ for any $1 \leq j \leq p$.

Finally, applying the same inductively we get:

$$\left(\sum_{i=1}^{k} X_i\right)^{p^{n+l-1}} = \left(\sum_{i=1}^{k} X_i^{p^n}\right)^{p^{l-1}} + p^l \cdot Z'$$

Then, the desired equality is shown because in the considered algebra $p^l \cdot Z' = 0$ □

Now we prove that abelian CA cannot randomize strongly, and cannot randomize some cylinders without randomizing in density.

THEOREM 3. *There is no strongly randomizing abelian CA with commuting endomorphisms. Moreover, for any group G, there exists $N \in \mathbb{N}$ such that, for any abelian CA F over G with commuting endomorphisms, the following are equivalent:*

(i) F randomizes in density;

(ii) $\mathrm{rank}\big(F^N(c)\big) \geq 2$ *for any finite configuration c of rank* 1*;*

(iii) F randomizes in density on cylinders of length 1*,*

where as usual the class of initial measures is the set of harmonically mixing measures.

Proof. By the decomposition theorem for finite abelian groups and using Corollary 1, it is enough to consider the case where G is a p-group (because an abelian CA on $G_1 \times G_2$ where G_1 and G_2 have relatively prime orders is a Cartesian product of abelian CAs on G_1 and G_2 respectively).

We can write F as usual

$$F = \sum_{i \in \mathcal{N}} \overline{\phi}_i \circ \sigma^i.$$

Consider the commutative ring generated by the $\overline{\phi}_i$ and the shift map under addition and composition. This ring has characteristic p^l for some l because we considered a p-group as the alphabet. By Lemma 9, we get

$$\forall n \in \mathbb{N},\ F^{p^{n+l-1}} = \left(\sum_{i\in\mathcal{N}} (\overline{\phi}_i)^{p^n} \circ \sigma^{ip^n}\right)^{p^{l-1}} = \sum_{j\in\mathcal{N}'} (\gamma_j)^{p^n} \circ \sigma^{jp^n}, \tag{3}$$

where

$$\mathcal{N}' = \{n_1 + \cdots + n_{p^{l-1}} : n_i \in \mathcal{N}\}$$

and each γ_j is a sum of compositions of some ϕ_i that do not depend on n. The number of terms in the right-hand expression is bounded independently of n, so the number of dependencies of F^t is bounded on an infinite sequence of times and so F cannot be strongly randomizing by Proposition 6 (it cannot even strongly randomize cylinders of size 1).

For the second part of the proposition, consider $N = p^{n_0+l-1}$ for some n_0 such that $p^{n_0} > |G|$. This choice of N guaranties that, for any endomorphism h of G, we have $\ker(h^{p^{n_0}}) = \ker(h^{p^n})$ for any $n \geq n_0$ (because the sequence $\ker(h^i)$ increases strictly until it stabilizes). We have the following alternative:

(a) *There is* c *of rank* 1 *such that* $\mathrm{rank}\big(F^N(c)\big) = 0$ (in particular, F is not surjective). Then, for any $t \geq N$ and any $n \geq 1$ $\left|\ker(F^t)\right|$ is at least 2^n when considering F restricted to periodic configurations of period n. By taking $n = |\mathcal{N}| \cdot t$ we deduce that $F^t\lambda([0]) \geq \frac{2}{|G|}$ for any $t \geq N$ so that F doesn't randomize cylinders of length 1 starting from the uniform Bernoulli measure.

(b) $\mathrm{rank}\big(F^N(c)\big) > 0$ *for any* c *of rank* 1 *but there is* d *of rank* 1 *such that* $\mathrm{rank}\big(F^N(d)\big) = 1$. By Equation 3, there is some $g \in G$ and a $j \in \mathcal{N}'$ such that $g \notin \ker(\gamma_j^{p^{n_0}})$ but $g \in \ker(\gamma_{j'}^{p^{n_0}})$ for all $j' \neq j$. As said before, the choice of n_0 ensures that for any $n \geq n_0$ we have $g \notin \ker(\gamma_j^{p^n})$ but $g \in \ker(\gamma_{j'}^{p^n})$ for all $j' \neq j$. Hence, using the formula for $F^{p^{n+l-1}}$, we deduce $\mathrm{rank}\big(F^{p^{n+l-1}}(d)\big) = 1$ for any $n \geq n_0$. In that case F admits a soliton of size 1.

(c) *For any* c *of rank* 1, $\mathrm{rank}\big(F^N(c)\big) \geq 2$. For the same reason as in the previous case we deduce $\mathrm{rank}\big(F^{p^{n+l-1}}(c)\big) \geq 2$ for any $n \geq n_0$. But the formula for $F^{p^{n+l-1}}$ shows that the non-zero cells in $F^{p^{n+l-1}}(c')$ belong to the set $\mathcal{N}'_n = \{jp^n : j \in \mathcal{N}'\}$ for any c' of rank 1. We deduce that, for any d of rank m and n large enough, $F^{p^{n+l-1}}(d)$ contains 2 non-zero cells distant from at least $p^n - m$ cells. Therefore F does not have any soliton and it randomizes harmonically mixing measures in density by Theorem 2.

Since (c) corresponds to (ii), we have shown that $(i) \Leftrightarrow (ii)$. Clearly $(i) \Rightarrow (iii)$, we prove that $(iii) \Rightarrow (c)$. If F randomizes in density on cylinders of length 1, then $\widehat{F}$ has no soliton of size 1. It is straightforward to check by Equation 2 that an abelian CA with commuting endomorphisms has a dual with commuting endomorphisms, and therefore the above alternative applies also to $\widehat{F}$. In other words, $\widehat{F}$ satisfies (c), which means that it admits no solitons. This means in turn that F must satisfy (c) as well. The Theorem follows. □

REMARK 3. *In [22] the authors consider abelian CA with integer coefficients (i.e. endomorphisms of the form $a \mapsto n \cdot a$); such a CA is called* proper *if for any prime divisor p of the order of the alphabet there are at least 2 coefficients not divisible by p. Theorem 6 of [22] shows that proper CA are randomizing in density. This is a particular case of the above theorem. Indeed, if F is proper and taking N from the theorem, it is easy to check that F^N is proper and that this is equivalent to* $\mathrm{rank}(F^N(c)) \geq 2$ *for any finite configuration of rank* 1.

6.2. *Strong Randomization* We now give examples of strongly randomizing abelian cellular automata. They are all defined over alphabet $\mathcal{A} = \mathbb{Z}_p^2$ where p is a prime number. In this section we denote by π_1 and π_2 the projections on the first and second component of the alphabet respectively.

To simplify notations, we denote by $\oplus$ the addition on $\mathcal{A}$, the componentwise addition on configurations of $\mathcal{A}^{\mathbb{Z}}$, and the addition of cellular automata over $\mathcal{A}^{\mathbb{Z}}$.

We also denote $n \cdot g = \underbrace{g \oplus \cdots \oplus g}_{n \text{ times}}$ and $\overline{0}$ the neutral element of the group $\left(\mathcal{A}^{\mathbb{Z}}, \oplus\right)$ (*i.e.* the configuration everywhere equal to $(0,0)$).

LEMMA 10. *If two abelian CA F and G over $\mathcal{A}^{\mathbb{Z}}$ commute, then for any $n \geq 0$ we have:*

$$(F \oplus G)^{p^n} = F^{p^n} \oplus G^{p^n}$$

Proof. Due to the structure of the group $\mathcal{A} = \mathbb{Z}_p^2$, for any configuration $c \in \mathcal{A}^{\mathbb{Z}}$ we have $p \cdot c = \overline{0}$. Therefore for any CA F, we have that $p \cdot F$ is the constant map equal to $\overline{0}$.

Now, from the binomial formula and from the fact that p divides $\binom{p}{n}$ for any $1 < n < p$ we deduce that:

$$(F \oplus G)^p = F^p \oplus G^p.$$

The lemma follows by an easy induction. □

Extending the example F_2, we now consider for each prime $p \geq 2$ and all $c \in \mathcal{A}^{\mathbb{Z}}$:

$$F_p(c)_z = \big(\pi_1(c_{z+1}) \oplus \pi_2(c_z), \pi_1(c_z)\big)$$
$$G_p(c)_z = \big(\pi_1(c_{z+1}) \oplus \pi_1(c_z) \oplus \pi_2(c_z), \pi_1(c_z)\big)$$

In the remainder of the section we prove that F_p and G_p are strongly diffusive for any prime $p \geq 2$.

These cellular automata are reversible and in fact also time-symmetric *i.e.* the product of two involutions [**10**]. For instance the inverse of F_p is:

$$F_p^{-1}(c)_z = \big(\pi_2(c_z), \pi_1(c_z) \oplus -\pi_2(c_{z+1})\big).$$

Since they are reversible, we extend their dependency diagrams to negative times, which means that $\Delta_\Phi(t,z)$ is defined for any $(t,z) \in \mathbb{Z}^2$ where Φ denotes F_p or G_p.

Note that both F_p and its inverse can be defined with neighborhood $\{0,1\}$. The same is true for G_p.

LEMMA 11. *For any $n \geq 0$, any $t \in \mathbb{Z}$ and any $z \in \mathbb{Z}$ we have:*

$$\Delta_{F_p}(2p^n + t, z) = \Delta_{F_p}(p^n + t, p^n + z) \oplus \Delta_{F_p}(t,z)$$
$$\Delta_{G_p}(2p^n + t, z) = \Delta_{G_p}(p^n + t, p^n + z) \oplus \Delta_{G_p}(p^n + t, z) \oplus \Delta_{G_p}(t,z)$$

Proof. First, it is straightforward to check that $F_p^2 = \sigma \circ F_p \oplus I$ (where I denotes the identity map over $\mathcal{A}^{\mathbb{Z}}$). Hence, using Lemma 10, we get the identity $F_p^{2p^n} = \sigma_{p^n} \circ F_p^{p^n} \oplus I$. For every configuration c, $t \in \mathbb{Z}$ and $z \in \mathbb{Z}$ we have $F_p^{2p^n+t}(c)_z = F_p^{p^n+t}(c)_{p^n+z} \oplus c_z$, which proves the Lemma.

The same proof scheme applies to Δ_{G_p}. □

LEMMA 12. *Let Φ be either F_p or G_p. For any $t \in \mathbb{Z}$ we have:*

1. *$\Delta_\Phi(t,z)$ is the constant map equal to $(0,0)$ when $z > 0$ or $z < -|t|$*
2. *$\Delta_\Phi(t,0)$ is a bijection.*

Proof. First both Φ and Φ^{-1} have neighborhood $\{0,1\}$. So the first item is straightforward by induction.

Second, by definition, $\Delta_\Phi(0,0)$ is a bijection. We can check that:

$$\Delta_{F_p}(1,0) : (g,h) \mapsto (h,g),\ \Delta_{G_p}(1,0) : (g,h) \mapsto (g{+}h,g),\ \Delta_{G_p}(2,0) : (g,h) \mapsto (h,g{+}h),$$

which are bijections. Applying Lemma 11 with $n = 0$, we get by straightforward induction that $\Delta_{F_p}(t+2,0) = \Delta_{F_p}(t,0)$ and $\Delta_{G_p}(t+3,0) = \Delta_{G_p}(t,0)$. We proved the second item. □

Much of the structure of Δ_Φ can be understood when focusing on particular "triangular" zones of $\mathbb{Z}^2$ at various scales. For $k \geq 0$ and n large enough so that $p^n - k > k$, we define the corresponding zone as:

$$T_{n,k} = \{(t,z) : k < t < p^n - k \text{ and } -t < z < k\}$$

LEMMA 13. *Let Φ be either F_p or G_p. Let $k \geq 0$ and n such that $p^n - k > k$. Then for any $(t,z) \in T_{n,k}$ and any $j \geq 1$ we have:*

$$\Delta_\Phi(t,z) = \Delta_\Phi(t + j \cdot p^n, z - j \cdot p^n).$$

In particular, if χ is a character whose support is of diameter at most k, then for any t with $k < t < p^n - k$ we have:

$$\operatorname{rank}(\chi \circ \Phi^{t+j\cdot p^n}) \geq \operatorname{rank}(\chi \circ \Phi^t) + 1$$

Proof. For the first assertion consider some $(t,z) \in T_{n,k}$. From Lemma 12, we have that both $\Delta_\Phi(t-p^n, z-p^n)$ and $\Delta_\Phi(t, z-p^n)$ are the constant map equal to $(0,0)$. Therefore the following identities obtained from Lemma 11

$$\Delta_{F_p}(t+p^n, z-p^n) = \Delta_{F_p}(t-p^n, z-p^n) \oplus \Delta_{F_p}(t,z)$$
$$\Delta_{G_p}(t+p^n, z-p^n) = \Delta_{G_p}(t-p^n, z-p^n) \oplus \Delta_{G_p}(t, z-p^n) \oplus \Delta_{G_p}(t,z)$$

can in both cases be simplified to:

$$\Delta_\Phi(t,z) = \Delta_\Phi(t+p^n, z-p^n).$$

This prove the case $j=1$. The same idea shows the induction step on j:

$$\Delta_\Phi(t+j\cdot p^n, z-j\cdot p^n) = \Delta_\Phi(t+(j+1)\cdot p^n, z-(j+1)\cdot p^n).$$

For the second assertion, apply Lemma 3 on the first assertion. We get that $-z \in \operatorname{supp}(\chi\circ\Phi^t)$ if and only if $-z+j\cdot p_n \in \operatorname{supp}(\chi\circ\Phi^{t+j\cdot p^n})$. Furthermore, we also have $0 \in \operatorname{supp}(\chi\circ\Phi^{t+j\cdot p^n})$ since 0 is a k-isolated bijective dependency by Lemma 12. Acconting for both contributions, we get $\operatorname{rank}(\chi\circ\Phi^{t+j\cdot p^n}) \geq \operatorname{rank}(\chi\circ\Phi^t)+1$. □

PROPOSITION 9. *F_p and G_p are strongly diffusive.*

Proof. Denote by Φ either F_p or G_p. Since Φ is reversible, there exists a constant C given by Lemma 2 such that for any $m \geq 0$, any $T \geq 0$, and any character of rank at least $C\cdot m\cdot T$, we have $\operatorname{rank}(F^t \circ \chi_0) \geq m$ for $1 \leq t \leq T$.

Let χ be any non-trivial character and let k be its diameter. Denote by $R(t) = \operatorname{rank}(\chi\circ\Phi^t)$. We are going to show that $R(t) \to \infty$, which implies the lemma since the choice of χ is arbitrary.

First, let n_0 be large enough and t_0 such that $k < t_0 < p^{n_0} - k$. By successive applications of Lemma 13, we get

$$R\big(t_0 + (p-1)p^{n_0} + (p-1)p^{n_0+1} + \cdots + (p-1)p^{n_0+m}\big) \geq m.$$

Since $p^{n_0+m+1} = \sum_{j=0}^m (p-1)p^{n_0+j} + p^{n_0}$ we deduce that

$$p^{n_0+m+1} - \big(t_0 + \sum_{j=0}^m (p-1)p^{n_0+j}\big) \leq p^{n_0} - t_0$$

so that as soon as $m \geq C\cdot M\cdot(p^{n_0}-t_0)$ it follows $R(p^{n_0+m+1}) \geq M$ by Lemma 2. Therefore:

$$R(p^n) \to_n \infty. \tag{4}$$

Now define the predicate $P_{t_0,n,m}$ as the conjunction of the following conditions:

1. $k \leq t_0 \leq p^n - k$;
2. $\forall t, t_0 \leq t \leq p^n - k : R(t) \geq m$;
3. $R(p^n - k) \geq C\cdot(m+1)\cdot(t_0+k)$.

First, since $R(t) \geq 1$ for any t (from Lemma 12), we have by Equation (4) above and Lemma 1 $P_{k,n,1}$ for n large enough.

Furthermore, if $P_{t_0,n,m}$, then $P_{t_0+p^n,n',m+1}$ for all large enough n'. Indeed, condition 1 is obviously true for $n' > n$, condition 3 is true for any large enough n' from (4) above and Lemma 1. Finally, condition 2 is obtained from Lemma 13: for any $j \geq 1$,

1. for $j \cdot p^n + t_0 \leq t \leq (j+1) \cdot p^n - k$, $R(t) \geq R(t - j \cdot p^n) + 1 \geq m + 1$;
2. for $(j+1) \cdot p^n - k \leq t \leq (j+1) \cdot p^n + t_0$ we have $R(t) \geq m + 1$ because $R((j+1) \cdot p^n - k) \geq R(p^n - k) \geq C \cdot (m+1) \cdot (t_0 + k)$.

We have shown that for any m there exists t_0 such that for any large enough n' we have $P_{t_0,n',m}$. This in particular implies $R(t) \geq m$ for all $t \geq t_0$. We conclude that $R(t) \to \infty$. □

6.3. *Randomizing only up to fixed-size cylinders* We now define a family of cellular automata that randomize finite cylinders up to a certain length, but no further. The alphabet is G^2 where G is any finite abelian group.

$$I_G(c)_z = \left([\pi_1(c_{z-1}) \oplus \pi_1(c_{z+1}) \oplus \pi_2(c_z)]^{-1}, \pi_1(c_z)\right)$$

Notice that $I_{\mathbb{Z}_2} = H_2$.

Proposition 10. *I_G randomizes cylinders of length 1, but not cylinders of length 2.*

Lemma 14. *For $t > 0$, we have:*

$$\Delta_{I_G}(t,z) = (g,h) \mapsto \begin{cases} 0 & \text{if} \quad |z| > t \\ (g^{(-1)^t}, 0) & \text{if} \quad |z| = t \\ (g^{(-1)^t}, h^{(-1)^{t+1}}) & \text{if} \quad |z| < t, t+z = 0 \mod 2 \\ (h^{(-1)^t}, g^{(-1)^{t+1}}) & \text{if} \quad |z| < t, t+z = 1 \mod 2 \end{cases}$$

Proof. By straightforward induction. □

Now we use Proposition 5 in conjunction with the following proposition to prove the announced result.

Proposition 11. *I_G is strongly χ-diffusive on characters of rank 1, but not on characters of length 2.*

Proof. Let χ be a character of rank 1, i.e. $\chi(x) = \chi_0(x_0)$ with $\chi_0 \neq 0$. For $t > 0$:

$$\chi \circ I_G^t(x) = \chi_0 \left(\bigotimes_{z=-t}^{t} \Delta_{I_G}(t,z)(x_{-z}) \right)$$

$$\prod_{z=-t}^{t} \chi_0 \circ \Delta_{I_G}(t,z)(x_{-z})$$

χ_0 is nontrivial and by the previous Lemma, $\Delta_{I_G}(t,z)$ is an isomorphism when $|z|<t$. We deduce that $\chi_0 \circ \Delta_{I_G}(t,z)$ is nontrivial whenever $|z|<t$ and therefore $\text{rank}(\chi \circ I_G^t(x)) \geq 2t-1 \to \infty$: I_G strongly χ-diffuses characters of rank 1.

For the second point, take any elementary character $\eta_0 \in \widehat{G}$ and define another character $\eta : x \mapsto \eta_0(\pi_1(x_0) \oplus \pi_2(x_1))$. Then, by a straightforward computation:

$$\begin{aligned}\eta \circ I_G(x) =& \eta_0\big([\pi_1(x_{-1}) \oplus \pi_1(x_1) \oplus \pi_2(x_0)]^{-1} \oplus \pi_1(x_1)\big)\\ =& \eta_0\big(\pi_1(x_{-1}) \oplus \pi_2(x_0)\big)^{-1} = \sigma^{-1} \circ \eta(x)^{-1}\end{aligned}$$

which is a soliton for $\widehat{F}$ of rank 2. □

Now we introduce the cellular automaton $I_{G,n}$, which consists in applying the local rule of I_G on the neighbourhood $\{z_{-n}, z_0, z_n\}$:

$$I_{G,n}(c)_z = \big([\pi_1(c_{z-n}) \oplus \pi_1(c_{z+n}) \oplus \pi_2(c_z)]^{-1}, \pi_1(c_z)\big)$$

Intuitively, a space-time diagram for $I_{G,n}$ consists of n intertwined space-time diagrams for I_G.

THEOREM 4. *Let $n \in \mathbb{N}$. $I_{G,n}$ randomizes cylinders of length n, but does not randomize cylinders of length $n+1$.*

It is obvious (straightforward induction) that:

LEMMA 15.
$$\Delta_{I_{G,n}}(t,z) = \begin{cases} \Delta_{I_G}(t, z/n) & \text{if } z = 0 \mod n \\ 0 & \text{otherwise} \end{cases}$$

so that we can use Lemma 14. As in the previous case, we use Proposition 5 in conjunction with the following:

PROPOSITION 12. *$I_{G,n}$ is strongly χ-diffusive on characters of support $\subset [0, n-1]$, but not on characters of support $\{0, n\}$.*

Proof. Let χ be a nonzero character of support $\subset [0, n-1]$, that is, $\chi = \prod_{i=0}^{n-1} \chi_i$; without loss of generality assume $\chi_0 \neq 0$. We have for any $t > 0$:

$$\begin{aligned}\chi \circ I_{G,n}^t(x) &= \prod_{i=0}^{n-1} \chi_i \left(\bigotimes_{z=-t}^{t} \Delta_{I_{G,n}}(t, z+i)(x_{-z}) \right)\\ &= \prod_{z=-t}^{t} \left(\prod_{i=0}^{n-1} \chi_i \circ \Delta_{I_{G,n}}(t, z+i) \right)(x_{-z})\end{aligned}$$

Now for any z such that $z = 0 \mod n$, the corresponding term in the previous equation is $\chi_0 \circ \Delta_{I_{G,n}}(t,z)$ by Lemma 15, and this term is an isomorphism when $t < z < t$ by Lemma 14. Therefore $\text{rank}(\chi \circ I_{G,n}^t) \geq 2\left\lfloor \frac{t}{n} \right\rfloor - 1 \to \infty$.

For the second point, take any nontrivial elementary character $\eta_0 \in \widehat{G}$ and define another character $\eta : x \mapsto \eta_0(\pi_1(x_0) \oplus \pi_2(x_n))$. This is a soliton for $\widehat{F}$ of support $\{0, n\}$ by Lemma 15 and the same proof as Proposition 11. □

7. *Open problems*

Building upon the approach of [**21, 22**] we completely characterized randomization in density for abelian CAs. Furthermore, we provided examples of other forms of randomization, most notably strong randomization (in simple convergence), that can only happen for abelian CA whose coefficient are noncommuting endomorphisms.

As mentioned by several authors, the most important research direction for randomization in CA is to develop tools and techniques to go beyond the abelian case:CA with a nonabelian group structure or nonlinear CA [**13, 14**]. There is some experimental evidence pointing at nonlinear randomization candidates [**12, 27**]. The class of bipermutive CA, althought it does not encompass all candidates, has some relevant related work: we know that the set of invariant measures is limited to the uniform Bernoulli measure in some cases [**1**] and that it exhibits a topological analogue to randomization [**25**].

We believe that several intermediate questions raised by the present work are worth being considered:

- Is strong randomization of an abelian CA equivalent to strong randomization of its dual?
- What is the importance of reversibility in the above examples of strong randomization? Our proof relies on reversibility and the smoothness it implies on the evolution of rank of characters. Can a reversible abelian CA be randomizing in density but not strongly randomizing?
- The notions of soliton and diffusivity can be defined for arbitrary CAs using diamonds (like in the notion of pre-injectivity) instead of finite configurations: what are the links with topological properties like positive expansivity, pre-expansivity†, mixing or transitivity? are there links with randomization beyond abelian CAs? can we generalize the characterization of randomization through solitons to non-abelian groups through representation theory?
- Theorem 3 gives a procedure to test randomization for abelian CA with commuting coefficients because the constant N can be explicitely computed. Is there an algorithm to decide the presence of a soliton in abelian CA? What about solitons in general CAs (again formalizing through diamonds instead of finite configurations)?

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† Pre-expansivity, introduced in [**11**], is the property of being positively expansive on diamonds (pairs of configurations with a finite set of differences). A reversible CA can be pre-expansive (but never positively expansive) like the example F_2 of this paper. Corollary 1 states that a direction of positive expansivity implies randomization in density: this implication actually holds for directions of pre-expansivity.

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