A note on directional closing

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Abstract

We show that directional closing in the sense of Guillon-Kari-Zinoviadis and Franks-Kra is not closed under conjugacy. This implies that being polygonal in the sense of Franks-Kra is not closed under conjugacy.

1 Introduction

Say a set $S \subset \mathbb{R}^2$ *codes* a set $T \subset \mathbb{R}^2$ for a subshift $X \subset A^{\mathbb{Z}^2}$ if

$$x, y \in X \wedge x|_{S \cap \mathbb{Z}^2} = y|_{S \cap \mathbb{Z}^2} \implies x|_{T \cap \mathbb{Z}^2} = y|_{T \cap \mathbb{Z}^2}.$$

A set S is *expansive* if it codes $\mathbb{R}^2$.

The following definitions of directional closing are due to Guillon, Kari and Zinoviadis [3] and John Franks and Bryna Kra [2] (with slightly different terminology).

Definition 1. *Let $H = \{(a,b) \in \mathbb{R}^2 \mid b < 0 \vee (b = 0 \wedge a < 0)\}$. A subshift $X \subset A^{\mathbb{Z}^2}$ is* right-closing in direction $\mathbf{v}$*, where $\mathbf{v} \in S^1$ is a direction represented by a point on the unit sphere, if $R(H)$ codes $\overline{R(H)}$, where R is the linear rotation that takes $(0,1)$ to $\mathbf{v}$. Left-closing is defined symmetrically, and a subshift is* closing *in direction $\mathbf{v}$ if it is either left- or right-closing in that direction, and* bi-closing *if it is both.*

Note that with this definition a subshift is closing in an irrational direction $\mathbf{v}$ if and only if it is *deterministic* in that direction, meaning the half-plane $R(H)$ is expansive. For rational directions, bi-closing does not imply determinism. For the spacetime subshift of a surjective CA, being directionally left-, right- or bi-closing in the opposite direction of where time flows agrees with the standard definition for cellular automata [5].

We mention the following lemma (not original to this note) which is one motivation for this definition. We say $T \subset \mathbb{Z}^2$ is an *extremally permutive* shape[1] for a subshift $X \subset A^{\mathbb{Z}^2}$ if $T \setminus \{\mathbf{v}\}$ codes T in X whenever $\mathbf{v}$ is an extremal point of the convex hull of T in $\mathbb{R}^2$.

Lemma 1. *If there exists an extremally permutive shape for a subshift X, then all directions of X are bi-closing.*

[1] For any two corners $\mathbf{u}, \mathbf{v}$, the contents of $T \setminus \{\mathbf{u}, \mathbf{v}\}$ determine a partial permutation between possible values at $\mathbf{u}$ and $\mathbf{v}$.

The converse is also true [3, 2].

Such shapes appear frequently in multidimensional symbolic dynamics and the theory of cellular automata, for example in algebraic symbolic dynamics [8], in the context of Nivat's conjecture [1, 4], and in the spacetime subshifts of bipermutive CA. The tiling problem stays undecidable for SFTs admitting $\{(0,0),(1,0),(0,1),(1,1)\}$ as a corner-deterministic shape [7].

In this note, we show that directional closing is (not surprisingly) not conjugacy-invariant. *Left* means $(-1,0)$.

Theorem 1. *The conjugacy class of the Ledrappier subshift contains a subshift that is neither left- nor right-closing to the left.*

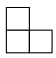
We only conjugate one direction to be non-closing, but readers familiar with marker techniques will have no trouble extending this to show that all three non-deterministic directions can be simultaneously conjugated to be non-closing (in both directions).

In a recent paper [2], Franks and Kra study subshifts admitting an extremally permutive shape. In their terminology, such subshifts are called *polygonal*. They ask in [2, Question 1] whether polygonal subshifts are closed under conjugacy. Our example above solves the question in the negative.

Theorem 2. *The conjugacy class of the Ledrappier subshift contains a non-polygonal subshift.*

2 Proofs

Proof of Theorem 1. Write $\mathbb{Z}_2$ for the ring $\mathbb{Z}/2\mathbb{Z}$. There is a standard way to see $\mathbb{Z}_2^{\mathbb{Z}^2}$ as a $\mathbb{Z}_2[\mathbf{x},\mathbf{x}^{-1},\mathbf{y},\mathbf{y}^{-1}]$-module, and the *Ledrappier subshift* $X \subset \mathbb{Z}_2^{\mathbb{Z}^2}$ is the variety $\{x \mid px = 0\}$ where $p = 1 + \mathbf{x} + \mathbf{y}$.

We fix the geometric convention that X is the subshift where looking through each window of shape  you see an even number of 1s.

We add another layer of information that implements the possibility of slight skewing of the location of bits, and show that the left direction can be made non-closing both left and right.

We define another subshift $Y \subset (\mathbb{Z}_2^2)^{\mathbb{Z}^2}$ where we allow either one or two bits per cell. We will define this as the image of X under an injective morphism $f : \mathbb{Z}_2^{\mathbb{Z}^2} \to (\mathbb{Z}_2^2)^{\mathbb{Z}^2}$, so in particular it conjugates X onto Y.

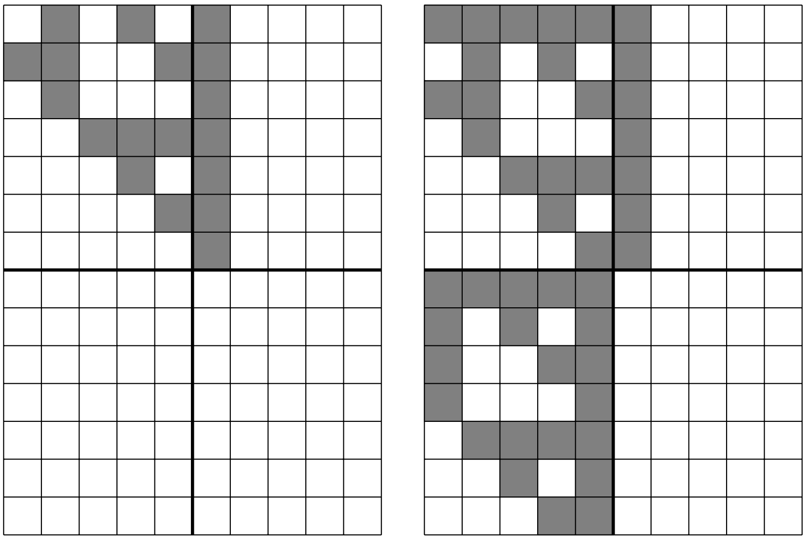
First define a one-dimensional injective morphism $g : \mathbb{Z}_2^{\mathbb{Z}} \to (\mathbb{Z}_2^2)^{\mathbb{Z}}$ by $g(x)_i = g_{\mathrm{loc}}(x_i, x_{i+1})$ where

$$g_{\mathrm{loc}}(a,b) = \begin{cases} (a,0) & \text{if } b = 0 \\ (0,1) & \text{if } b = 1 \end{cases}$$

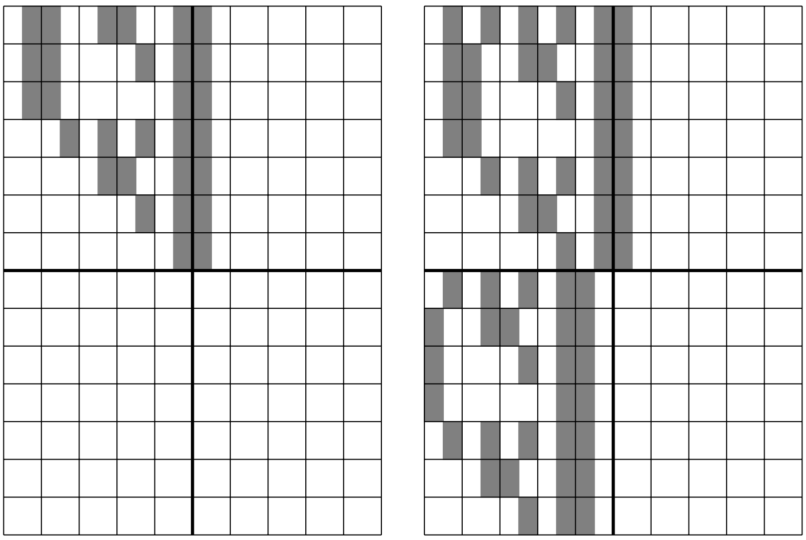
This is just the 2-blocking representation [6] followed by the symbolwise projection that maps $(1,1)$ to $(0,1)$ and fixes other symbols. A left inverse is obtained by projecting to the second coordinate and shifting, so g is indeed injective.

We define f by applying g on every row, or in formulas $f(x)_{\vec{v}} = g(x_{\vec{v}_2})_{\vec{v}_1}$ where for $x \in \mathbb{Z}_2^{\mathbb{Z}^2}$ we write $x_i \in \mathbb{Z}_2^{\mathbb{Z}}$ for row extraction, i.e. $(x_i)_j = x_{(j,i)}$.

Now consider the (obvious infinite extensions of the) following configurations



where the color gray denotes 1. The conjugate images in Y are (respectively)



where we color the left and right half of each cell gray or white depending on the left- and rightmost bit, respectively, again gray means 1. The picture plainly shows that Y is not left-closing to the left. Similarly one can show it is not right-closing to the left. $\square$

Proof of Theorem 2. By Lemma 1, a polygonal subshift is bi-closing in all directions. Thus, the subshift Y constructed in the previous proof is not polygonal. $\square$

References

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