Asymptotic Nilpotency Implies Nilpotency in Cellular Automata on the d-Dimensional Full Shift ⋆

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Abstract. We prove a conjecture in [3] by showing that cellular automata that eventually fix all cells to a fixed symbol 0 are nilpotent on $S^{\mathbb{Z}^d}$ for all d. We also briefly discuss nilpotency on other subshifts, and show that weak nilpotency implies nilpotency in all subshifts and all dimensions, since we do not know a published reference for this.

Keywords: cellular automata, subshifts, asymptotic nilpotency, nilpotency

1 Introduction

One of the most interesting aspects in the theory of cellular automata is the study of different types of nilpotency, that is, different ways in which a cellular automaton can force a particular symbol (usually called 0) to appear frequently in all its spacetime diagrams. The simplest such notion, called simply 'nilpotency', is that the cellular automaton maps every configuration to a uniform configuration $\ldots 000 \ldots$, on which it behaves as the identity, in a uniformly bounded number of steps, that is, c^n is a constant map for some n. The notion of 'weak nilpotency', where for all $x \in X$, we have $c^n(x) = \ldots 000 \ldots$ for some n, is equivalent to nilpotency in all subshifts $X \subset S^{\mathbb{Z}^d}$. We don't know a published reference for this, and give a proof in Proposition 1. There are at least two ways to define variants of nilpotency using measure theory, and one, called 'ergodicity', is shown to be equivalent to nilpotency in [1], while the notion of 'unique ergodicity' is shown to be stricly weaker in [9].

A particularly nice variant of nilpotency is the 'asymptotic nilpotency' defined and investigated in at least [4] (where asymptotically nilpotent cellular automata are called 'cellular automata with a nilpotent trace') and [3]. This notion is shown to be equivalent to nilpotency on one-dimensional full shifts in [4], and on one-dimensional transitive SFTs in [3], but interestingly the proofs are not trivial, unlike for most of the notions listed above which coincide with nilpotency. It was left open in [4] whether asymptotic nilpotency implies nilpotency in all dimensions, that is, on the groups $\mathbb{Z}^d$ for arbitrary d, and the proof

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given in [4] does not generalize above $d = 1$ as such. The problem is restated as open in [3]. In this article, we prove that asymptotic nilpotency indeed implies nilpotency in Theorem 2 by reducing to the one-dimensional case, and ask whether this holds in all groups.

The study of nilpotency properties of cellular automata belongs to the more general study of asymptotic behavior of cellular automata, where emphasis is usually put on the limit set of a cellular automaton, see [6] for a survey on this topic. More generally, the study of cellular automata is subsumed by the study of multidimensional SFTs, in that a d-dimensional cellular automaton can be though of as a $(d+1)$-dimensional SFT deterministic in one dimension. See [8] for a thorough investigation into the 'limit behavior' of these systems.

We also slightly extend the result of [3] that asymptotic nilpotency implies nilpotency on one-dimensional transitive SFTs by removing the assumption of transitivity, and using this observation obtain some more cases in which asymptotic nilpotency implies nilpotency on multidimensional SFTs in Theorem 4. We also give two simple examples of one-dimensional sofic shifts where asymptotic nilpotency does not imply nilpotency. Finally, we ask whether *any* two-dimensional SFT can admit an asymptotically nilpotent but not nilpotent cellular automaton.

2 Definitions and Initial Observations

Let S be a finite set called the *alphabet*, whose elements we call *symbols*. The set $S^{\mathbb{Z}^d}$ with the product topology induced by the discrete topology of S is called the *d-dimensional full shift*. We call the elements $x \in S^{\mathbb{Z}^d}$ *configurations*, and for $s \in S$, we write $s^{\mathbb{Z}^d}$ for the *all-s configuration* x such that $x_v = s$ for all $v \in \mathbb{Z}^d$. We denote by $\sigma^v : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ the *shift action* defined by $\sigma^v(x)_u = x_{u+v}$. The closed sets which are invariant under all σ^v are called *subshifts*. Alternatively, they can be defined by a set of forbidden patterns [7], and when this set can be taken to be finite, we say X is a subshift of finite type, or an *SFT*. For a subshift $X \subset S^{\mathbb{Z}^d}$, a continuous function $c : X \to X$ is called a *cellular automaton* if it commutes with the maps σ^v. Of course, the shift actions σ^v are themselves cellular automata. Again, cellular automata have a combinatorial description as functions induced by a local rule [5]. That is, if c is a cellular automaton, there is a finite set of vectors V such that $c(x)_u$ only depends on the pattern x_{u+V}. The quantity $r = \max_{v \in V} |v|$ is called the *radius* of c, where $|\cdot|$ is the norm $|v| = \max_i |v_i|$.

For a one-dimensional subshift X, we denote by $\mathcal{L}(X)$ the set of words that appear in configurations of X, called the *language* of X. We denote by $\mathcal{L}^{-1}(L)$ the closure of the set of configurations whose subwords are words in $\mathrm{fact}(L)$, where $\mathrm{fact}(L)$ is the closure of L under taking subwords. The set $\mathcal{L}^{-1}(L)$ is a subshift for all languages L, and when $\mathcal{L}(X) = L$ for a subshift X, we have $\mathcal{L}^{-1}(L) = X$. We say a one-dimensional subshift X is *sofic* when $\mathcal{L}(X)$ is a regular language, and in general, a d-dimensional subshift is called sofic when it is the projection of an SFT of dimension d by a pointwise symbol-to-symbol map. When $d = 1$, these definitions coincide [7].

A one-dimensional subshift X is said to be *transitive* if

$$u, v \in \mathcal{L}(X) \implies (\exists w)(uwv \in \mathcal{L}(X)),$$

and X is said to be *mixing* if

$$u, v \in \mathcal{L}(X) \implies (\forall_\infty n)(\exists w)(|w| = n \wedge uwv \in \mathcal{L}(X)),$$

where $\forall_\infty$ is the 'for all large enough' quantifier. That is, transitivity implies that any two words that occur in the subshift can be glued together with some word w, and mixing implies that the length can be chosen freely provided it is sufficiently large. We will also need the decomposition of a one-dimensional sofic shift into its transitive components in Corollary 1, and refer to [7] for the definitions and proofs.

Let $c : X \to X$ be a cellular automaton on a subshift $X \subset S^{\mathbb{Z}^d}$. We say c is 0-*nilpotent* if

$$(\forall_\infty n)(\forall x \in X)(\forall v \in \mathbb{Z}^d)(c^n(x)_v = 0).$$

The *limit set* of c is the set of configurations having infinitely long preimage chains, and it is well-known that c is s-nilpotent for some $s \in S$ if and only if its limit set is a singleton [2]. We say c is *weakly* 0-*nilpotent* if

$$(\forall x \in X)(\forall_\infty n)(\forall v \in \mathbb{Z}^d)(c^n(x)_v = 0),$$

and we say c is *asymptotically* 0-*nilpotent* if

$$(\forall x \in X)(\forall v \in \mathbb{Z}^d)(\forall_\infty n)(c^n(x)_v = 0).$$

It is clear from these formulas that nilpotency implies weak nilpotency, which in turn implies asymptotic nilpotency. In this article, we prove that the first implication is always an equivalence, and the second is an equivalence at least when $X = S^{\mathbb{Z}^d}$ (see Theorem 4 for the exact cases we are able to prove). A symbol s is said to be *quiescent* for c if $c(s^{\mathbb{Z}^d}) = s^{\mathbb{Z}^d}$. If c is 0-nilpotent, weakly 0-nilpotent or asymptotically 0-nilpotent, then 0 must be a quiescent state. Usually, the symbol 0 is clear from the context and is omitted.

A configuration x is said to be 0-*finite* if it has only finitely many non-0 symbols. It is called 0-*mortal for* c if $c^n(x) = 0^{\mathbb{Z}^d}$ for large enough n. Note that weak 0-nilpotency is equivalent to every configuration being 0-mortal. Given a configuration x, its *trace at the origin* is the one-way infinite word $(c^n(x)_{\mathbf{0}})_{n \in \mathbb{N}}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^d$, and its *nonzero trace at the origin* is the set $\{n \in \mathbb{N} \mid c^n(x)_{\mathbf{0}} \neq 0\}$. Asymptotic 0-nilpotency is then equivalent to all configurations having a finite nonzero trace at the origin. For $x \in S^{\mathbb{Z}^d}$, we write $\mathrm{supp}_0(x) = \{v \in \mathbb{Z}^d \mid x_v \neq 0\}$ for the *support* of x, so a 0-finite configuration is just a configuration with finite support. For $x, y \in S^{\mathbb{Z}^d}$ with $\mathrm{supp}_0(x) \cap \mathrm{supp}_0(y) = \emptyset$, we define

$$(x +_0 y)_v = \begin{cases} x_v, \text{ if } x_v \neq 0, \\ y_v, \text{ if } y_v \neq 0, \\ 0, \quad \text{otherwise.} \end{cases}$$

Again, in all these definitions, both 0 and c are omitted if they are clear from context.

For a subshift $X \subset S^{\mathbb{Z}^d}$ and $v \in \mathbb{Z}^d$, we denote by $X(v)$ the subshift $\{x \in X \mid \sigma^v(x) = x\}$. When $v^j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is a standard basis vector and $1 \leq p \in \mathbb{N}$, we define the map $\phi = \phi_{j,p} : X(pv^j) \to (S^p)^{d-1}$ by $\phi(x)_u = x_{u(0)} x_{u(1)} \cdots x_{u(p-1)}$ where $u(i) = (u_1, \ldots, u_{j-1}, i, u_j, \ldots u_d)$. Clearly, ϕ is a homeomorphism between $X(pv^j)$ and $Y = \phi(X(pv^j))$, and we obtain a $(d-1)$-dimensional action $c_{j,p} : Y \to Y$ by defining $c_{j,p}(\phi(x)) = \phi(c(x))$. This is well-defined because ϕ is bijective, and it is continuous and shift-commuting because ϕ, ϕ^{-1} and c are, so $c_{j,p}$ is a cellular automaton (although some small technical care needs to be taken in showing this due to the coordinate shift at j). Moreover, it is easy to see that $c_{j,p}$ is nilpotent (asymptotically nilpotent) if and only if c is nilpotent (asymptotically nilpotent) on $X(pv^j)$.

For a vector $v \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, we define $B_k(v) = \{u \mid |u - v| \leq k\}$ where again $|\cdot|$ is the norm $|v| = \max_i |v_i|$, and for a set of vectors $V \subset \mathbb{Z}^d$ and $k \in \mathbb{N}$, we define $B_k(V) = \bigcup_{v \in V} B_k(v)$. For $v \in \mathbb{Z}^d$ and $j, k \in \mathbb{N}$, we define

$$\mathrm{tower}(j, k, v) = B_k(\{u \in \mathbb{Z}^d \mid \forall i \neq j : u_i = v_i\}) \subset \mathbb{Z}^d.$$

This is a tower of width k around the vector v, extending in the directions v^j and $-v^j$. Finally, for a set of vectors $V \subset \mathbb{Z}^d$ and $j, k \in \mathbb{N}$, we analogously define

$$\mathrm{tower}(j, k, V) = \bigcup_{v \in V} \mathrm{tower}(j, k, v).$$

3 The Results

First, we show that weak nilpotency implies (and is thus equivalent to) nilpotency in all subshifts $X \subset S^{\mathbb{Z}^d}$.

Proposition 1. *Let $X \subset S^{\mathbb{Z}^d}$ be a subshift. Then a cellular automaton c on X is nilpotent if and only if it is weakly nilpotent.*

Proof. Suppose on the contrary that the CA $c : X \to X$ is weakly nilpotent but not nilpotent for some subshift $X \subset S^{\mathbb{Z}^d}$. Then there exists a configuration x in the limit set of c with $x_{\mathbf{0}} \neq 0$, for the symbol 0 such that all configurations reach the all-0 configuration in finitely many steps. Since x is in the limit set, it has an infinite chain $(x^i)_{i \in \mathbb{N}}$ of preimages. If r is the radius of c, then since 0 is a quiescent state, there must exist a sequence of vectors $(v^i)_{i \in \mathbb{N}}$ such that $|v^i - v^{i+1}| \leq r$ and $\sigma^{v^i}(x^i)_{\mathbf{0}} \neq 0$ for all $i \in \mathbb{N}$.

Let y be a limit of a converging subsequence of $(\sigma^{v^i}(x^i))_{i \in \mathbb{N}}$ in the product topology of $S^{\mathbb{Z}^d}$. Since c is weakly nilpotent, there exists $n \in \mathbb{N}$ such that $c^n(y) = 0^{\mathbb{Z}^d}$. Let $i \in \mathbb{N}$ be such that

$$y_{B_{2rn}(\mathbf{0})} = \sigma^{v^{i+n}}(x^{i+n})_{B_{2rn}(\mathbf{0})}. \tag{1}$$

By definition of the x^i and v^i, we have that $c^n(x^{i+n}) = x^i$ and thus

$$c^n(y)_{B_{rn}(\mathbf{0})} = \sigma^{v^{i+n}}(x^i)_{B_{rn}(\mathbf{0})}$$

contains a nonzero symbol, a contradiction, since we assumed $c^n(y) = 0^{\mathbb{Z}^d}$.

This naturally generalizes for cellular automata on subshifts of S^G where G is any group generated by a finite set A, by using the distance $d(g,h) = \min\{n \mid \exists g_1, \ldots, g_n \in A : g^{-1}h = g_1 \ldots, g_n\}$.

We repeat Theorem 3 of [4] and show how to apply it to prove Conjecture 1 of [3] in Theorem 2.

Theorem 1 (Theorem 3 of [4]). *Asymptotic nilpotency implies nilpotency for cellular automata on* $S^{\mathbb{Z}}$.

In fact, the following results were already explicitly proved in [4] on full shifts in all dimensions. We will briefly outline the proof of Lemma 2 for completeness. Lemma 1 is proven with a direct compactness argument.

Lemma 1. *If* $X \subset S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ *is a subshift and the cellular automaton* $c : X \to X$ *is asymptotically nilpotent, then*

$$(\forall k)(\exists n)(\forall x \in X)(\exists 0 \leq j \leq n)(\forall v \in B_k(\mathbf{0}))(c^j(x)_v = 0).$$

That is, for all k, there is a uniform bound n for the first time the cells $B_k(\mathbf{0})$ are simultaneously zero.

Lemma 2. *If* $X \subset S^{\mathbb{Z}^d}$ *is an SFT with dense finite points, the cellular automaton* $c : X \to X$ *is asymptotically nilpotent, and all finite patterns are mortal for* c*, then* c *is nilpotent.*

Proof. If c is not nilpotent, then for any k, there exists a configuration z_k where the nonzero trace at the origin contains a number larger than k. We may assume the z_k are finite by the assumption that finite points are dense, and they are then automatically mortal. By Lemma 1, the z_k can further be taken to have no nonzero values in the coordinates $B_k(\mathbf{0})$. Such finite configurations can then be stacked around each other (by the fact X is an SFT) to obtain a configuration whose nonzero trace at the origin is infinite, and thus c cannot be asymptotically nilpotent. □

In particular, the previous lemma holds for the full shift. The generalization to subshifts with dense finite points is needed for the analogous generalization of Theorem 2 to Theorem 4.

Theorem 2. *Asymptotic nilpotency implies nilpotency for cellular automata on* $S^{\mathbb{Z}^d}$.

Proof. By Theorem 1, we know that asymptotic nilpotency implies nilpotency in the case $d = 1$. We will prove the case $d = 2$, and informally explain how the general case is proved. So let $c : S^{\mathbb{Z}^2} \to S^{\mathbb{Z}^2}$ be asymptotically nilpotent with radius r. Let further $X = S^{\mathbb{Z}^2}$ and let $v^1 = (1, 0)$ and $v^2 = (0, 1)$ be the standard basis vectors of $\mathbb{Z}^2$. First, we note that c is nilpotent on $X(pv^2)$ for all p: the one-dimensional CA $c_{2,p}$ is asymptotically nilpotent on the one-dimensional full shift $\phi_{2,p}(X(pv^2)) = (S^p)^{\mathbb{Z}}$, and thus nilpotent by Theorem 1, implying that c is nilpotent on $X(pv^2)$.

Now, the basis of the proof is the following observation: given any finite configuration, if we 'add a vertical period', it becomes mortal. That is, let x be a finite configuration, and let $\mathrm{supp}(x) \subset B_k((0, 0))$ for some k. Then, if $m \geq 2k+1$, the configuration $\sum_{i\in\mathbb{Z}} \sigma^{imv^2}(x) \in S^{\mathbb{Z}^d}$ is well-defined, and it is mortal because it is vertically periodic, by the argument of the previous paragraph. Then, if there exist finite patterns extending arbitrarily far in the directions spanned by v^1, we can use argumentation similar to that of [4] with 'horizontally finite' points (that is, vertically periodic points which use only finitely many columns) to find a contradiction to asymptotic nilpotency.

Let us make this more precise. We first claim that there exists k such that for all finite configurations x, we have

$$\bigcup_{i\in\mathbb{N}} \mathrm{supp}(c^i(x)) \subset \mathrm{tower}(2, k, \mathrm{supp}(x)). \tag{2}$$

We show that if this does not hold, we can construct a configuration that contradicts asymptotic nilpotency, so assume that for all k, the finite configuration $x^k \in S^{\mathbb{Z}^2}$ is a counterexample to (2) for k, and for all k, let

$$u^k \in (\bigcup_{i\in\mathbb{N}} \mathrm{supp}(c^i(x^k))) \setminus \mathrm{tower}(2, k, \mathrm{supp}(x^k)).$$

Using the hypothetical x^k and u^k, we inductively construct a configuration x where $c^j(x)_{(0,0)} \neq 0$ for arbitrarily large j. Let y^0 be the all zero configuration, and take as the induction hypothesis that

- y^i is a mortal vertically periodic configuration,
- y^i has all of its nonzero cells in a finite amount of columns,
- the nonzero trace of y^i at the origin is a proper superset of that of y^{i-1}, if $i > 0$.

Now, let us construct y^{i+1} assuming y^i satisfies the induction hypothesis. Let m be such that y^i becomes zero in at most $m + 1$ steps, and that $\mathrm{supp}(c^n(y^i)) \subset \mathrm{tower}(2, m, (0, 0))$ for all n. Such m exists because y^i is mortal and has its nonzero cells in finitely many columns. Then for $k = (r + 1)m + 2r + 1$, we have that for $y^i + \sigma^{u^k}(x^k)$, two nonzero cells arising from y^i and $\sigma^{u^k}(x^k)$, respectively, are never seen in the same neighborhood, since the configuration y^i dies before it is reached by the nonzero cells evolving from $\sigma^{u^k}(x^k)$. More precisely, we have

$$c^j(y^i + \sigma^{u^k}(x^k)) = c^j(y^i) + c^j(\sigma^{u^k}(x^k)),$$

for all $j \in \mathbb{N}$.

We thus see that, at the origin, the nonzero values arising from y^i are followed by those arising from $\sigma^{u^k}(x^k)$ in the evolution of c, and by the assumption on x^k and u^k, the nonzero trace at the origin increases in cardinality by at least 1. By adding a vertical period for $\sigma^{u^k}(x^k)$, obtaining a configuration y, we see that $y^{i+1} = y^i + y$ is mortal. Furthermore, if the period of y is chosen large enough that the relevant initial part of the trace is not changed, the nonzero trace of y^{i+1} is a proper superset of that of y^i, and it thus satisfies the induction hypothesis. The configuration $x = \lim_i y^i$ now contradicts asymptotic nilpotency.

This means that no finite configuration can extend arbitrarily far in the horizontal directions, and with an analogous proof we see that no finite configuration can extend arbitrarily far vertically either. It is now easy to see that every finite configuration is in fact mortal, and Lemma 2 concludes the proof.

Now, consider the general case $d > 1$. We can prove this in two ways, either reducing directly to the case $d = 1$ or proceeding by induction on d. To reduce to the case $d-1$, we take any basis vector v and, analogously to the case $d = 2$, prove that a finite pattern can only expand arbitrarily far in the directions spanned by v, since asymptotic nilpotency implies nilpotency for full shifts in dimension $d-1$, and our previous argument can be used with the y^i only using a finite amount of $(d-1)$-dimensional hyperplanes. Formally, this can be done using the homeomorphisms $\phi_{j,p}$ where $v = v_j$. Now, a finite configuration cannot expand arbitrarily far in *any* direction (since towers given by any two distinct base vectors have a finite intersection), and Lemma 2 applies.

To reduce to the case $d = 1$ (Theorem 1) directly, we can, for all basis vectors v, extend finite patterns into configurations with $d-1$ (that is, all but v) directions of periodicity. Then, running c on such a configuration simulates a one-dimensional asymptotically nilpotent cellular automaton, which is then nilpotent, and we can show using our previous argument that a finite pattern cannot extend arbitrarily far in the directions spanned by v. Now, going through all basis vectors v, we see that finite configurations are mortal, and Lemma 2 again applies. □

It is hard to imagine a group in which asymptotic nilpotency does not imply nilpotency, leading to the obvious question:

Question 1. Does asymptotic nilpotency imply nilpotency on full shifts in all finitely generated groups?

When the full shift is replaced by an arbitrary subshift, this is easily seen not to be the case. For example, asymptotic nilpotency does not imply nilpotency on all one-dimensional sofic shifts: a nontrivial shift action is asymptotically nilpotent but not nilpotent on $\mathcal{L}^{-1}(0^*10^*)$. In fact, asymptotic nilpotency doesn't even necessarily imply nilpotency if the sofic shift is mixing:

Example 1. Let X be the mixing sofic shift $\mathcal{L}^{-1}((0^*l0^*r)^*)$ and let $c : X \to X$ be the cellular automaton that moves l to the left and r to the right, removing l and r when they collide. Clearly, c is asymptotically nilpotent but not nilpotent.

We can, however, prove that asymptotic nilpotency implies nilpotency on all SFTs in dimension one, as a corollary of a theorem in [3].

Theorem 3 (Theorem 4 of [3]). *Asymptotic nilpotency implies nilpotency on transitive one-dimensional SFTs.*

Corollary 1. *Asymptotic nilpotency implies nilpotency on all one-dimensional SFTs.*

Proof. Let $X \subset S^{\mathbb{Z}}$ be an SFT and let $c : X \to X$ be asymptotically nilpotent. On each transitive component Y of X, c is in fact nilpotent by Theorem 3. Let n be such that all the finitely many transitive components of X are mapped to $0^{\mathbb{Z}}$ by c^n. Now, consider an arbitrary configuration $x \in X$. The configuration $c^n(x) \in X$ must be both left and right asymptotic to $0^{\mathbb{Z}}$, that is, $c^n(x)_i = 0$ if $|i|$ is large enough, so by the assumption that X is an SFT, $c^n(x) \in Y$ for some transitive component Y of X. But this means $c^{2n}(x) = 0^{\mathbb{Z}}$. □

Of course, sofic shifts where asymptotic nilpotency does not imply nilpotency exist in any dimension, since they exist in dimension one. However, the case of an SFT seems harder due to the surprisingly complicated nature of multidimensional SFTs. However, if finite configurations are dense in a d-dimensional SFT, the proof of Theorem 2 works rather directly using Corollary 1 and Lemma 2.

Theorem 4. *If $X \subset S^{\mathbb{Z}^d}$ is a subshift where finite points are dense and $c : X \to X$ is an asymptotically nilpotent cellular automaton, then c is nilpotent.*

Proof. The case $d = 1$ follows from Corollary 1. We again only explicitly consider the case $d = 2$, and the proof for $d > 2$ is obtained as in Theorem 2. First, we note that the subshift $X(pv^2)$ is mapped through $\phi_{2,p}$ into a 1-dimensional SFT, so Corollary 1 applies. Now, if $x \in X$ is a finite point, the sum $\sum_{i \in \mathbb{Z}} \sigma^{imv^2}(x)$ is defined for all large enough m, since X is an SFT. This means that we can prove, using the arguments of the proof of Theorem 2, that finite points of X cannot expand arbitrarily far horizontally or vertically, which implies they are mortal. Then, Lemma 2 implies that c is nilpotent. □

Question 2. Does asymptotic nilpotency imply nilpotency in all two-dimensional SFTs?

More generally, it would be interesting to see what could be done in the more general framework of projective subdynamics, where the cellular automata are – in a sense – nondeterministic. In particular, it would be interesting to see whether these techniques could help in generalizing Theorem 6.4 of [8] to dimensions higher than 2.

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